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Perturbation theory of parametrically driven capillary waves at low viscosity

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We present a critical review of the Hamiltonian and the Lagrangian theories of pattern formation in driven capillary waves at low viscosity and high aspect ratio. We construct a Hamiltonian perturbation theory in the spirit of Milner's (1991) formulation, and derive the amplitude equations and their coefficients relevant at the onset of surface waves. Our presentation is detailed, and we carefully point out the differences between our results for the nonlinear coefficients and the results obtained by others. From our standing wave analysis we find that the square pattern is subcritical. Among the supercritical standing wave patterns, we find that the eightfold quasi-crystalline pattern, observed by Christiansen *et al.* (1992) and by Bosch (1995), is more stable than both rolls and hexagons. We outline the high-aspect-ratio experimental results obtained so far, and discuss them in the light of the theory.

1. Introduction

The generation of surface waves on a fluid subjected to purely vertical vibrations has a long and interesting history. In the summer of 1831, Faraday (1831) performed a number of experiments with liquids on a vibrating plate. He particularly noted that the frequency of the waves formed was half the frequency of the support, an observation that was confirmed in an electromagnetically controlled experiment by Rayleigh (1883) more than 50 years later. Rayleigh recognized that the instability at which surface waves are formed (now known as the Faraday instability) is of the parametric type, and he analysed Mathieu's equation in order to find a necessary condition for the subharmonic response. However, the subharmonic nature of the instability was first theoretically verified directly from the hydrodynamical equations by Benjamin & Ursell (1954) 70 years later. With the recent increased interest in nonlinear dynamics, much attention has again been devoted to surface waves, experimentally as well as theoretically (see e.g. Zakharov, L'vov & Falkovich 1992; Cross & Hohenberg 1993; Kudrolli & Gollub 1996; and references therein).

One of the intriguing observations near the Faraday instability is the selection of patterns that can be decomposed into a limited set of plane waves, with amplitudes a_j, wave vectors $\boldsymbol{k}_j$ of length k, and angular frequency $\omega = \omega_e/2$, where ω_e is the angular frequency of the external forcing. This paper is concerned with the derivation of a consistent set of amplitude equations describing the evolution of patterns generated above the Faraday instability in large-aspect-ratio driven capillary surface waves in

low-viscosity fluids. By low viscosity we mean $\nu k^2 \ll \omega$, where ν is the kinematic viscosity. Situations where this condition is not fulfilled has been considered for example by Edwards & Fauve (1994), Kumar & Tuckerman (1994), Bechhoefer *et al.* (1995), Beyer & Friedrich (1995), Kumar (1996), and Müller *et al.* (1997).

The equations we derive rest on the condition $kl_c \gg 1$, where $l_c^2 = 2\sigma/\rho g$ is the capillary length of the fluid, expressed in terms of the surface tension, σ, and the density, ρ, of the fluid, and the gravitational acceleration, g. This condition ensures that the surface waves are capillary waves, not gravitational waves. Moreover, the equations rely on the condition $kl \gg 1$, where l is a characteristic lateral dimension of the system. This is our working definition of a large-aspect-ratio system, although other relevant length scales must be taken into account (Edwards & Fauve 1994), as we shall discuss in §6.2. Finally, we assume that $kh \gg 1$, where h is the fluid depth. The calculations to be derived will be correct to $O(\nu)$ and to $O(\epsilon^{3/2})$, where the dimensionless control parameter ϵ is taken as $\epsilon = k(\bar{f} - \bar{f}_c)$, where $\bar{f}$ is the amplitude of the external drive, $\bar{f}\cos(\omega_e t)$, and $\bar{f}_c$ is the critical value of $\bar{f}$ at which capillary waves are formed. In the absence of damping, $\bar{f}_c = 0$ and $\epsilon = k\bar{f}$.

Efforts to derive amplitude equations, or similar equations, for capillary waves from first principles have recently been presented by several authors in a number of important contributions (Zakharov, L'vov & Starobinets 1971; Zakharov *et al.* 1992; Ezerskii *et al.* 1986; Milner 1991; Miles 1976, 1977, 1984, 1992, 1993, 1994; Umeki & Kambe 1989, Kambe & Umeki 1990; Umeki 1991, 1996; Zhang & Viñals 1996, 1997). However, the attempts to compare the results of the various calculations have not led to a unique description, and the efforts to detect the discrepancies in the various formulations have not been very successful. It is therefore an important issue in this paper to give a detailed derivation of the amplitude equations that allows others to follow (and check) the theory without excessive efforts.

A number of high-aspect-ratio low-viscosity experiments on capillary waves, where the above-mentioned theories may be relevant, have been carried out (Ezerskii, Korotin & Rabinovich 1985; Ezerskii *et al.* 1986; Ezersky *et al.* 1994; Levin & Trubnikov 1986; Tufillaro, Ramshankar & Gollub 1989; Christiansen, Alstrøm & Levinsen 1992, 1995; Bosch & van de Water 1993; Bosch 1995; Kudrolli & Gollub 1996; Binks & van de Water 1997). From these experiments it seems that the square pattern is the preferred pattern at aspect ratios below approximately 40–50, while new patterns in the form of a quasi-crystalline and a hexagonal pattern are formed at larger aspect ratios (Christiansen *et al.* 1992; Bosch 1995). We shall discuss this further in §6. There has also been focus on the shape of eigenmodes and mode competition (Ciliberto & Gollub 1984, 1985; Douady & Fauve 1988; Simonelli & Gollub 1989; Douady 1990; Christiansen *et al.* 1995). This is however outside the scope of this paper (for a review, see e.g. Miles & Henderson 1990). Furthermore, we shall only consider single-frequency forcing (for multi-frequency forcing, see Edwards & Fauve 1992, 1993, 1994; Müller 1993).

In §2, we first review the Hamiltonian and Lagrangian formulations of a nonlinear theory describing capillary waves. The relevant amplitude equations to be derived are those for travelling waves in the absence and presence of damping, and an amplitude equation for standing waves. The derivation of the amplitude equations is based on a multiple-scales expansion of the basic hydrodynamical or Hamiltonian equations describing capillary waves (Milner 1991 (§2.1), Zhang & Viñals 1996, 1997 (§2.2)), or on an equivalent, scaled expansion due to Miles (1976, 1977, 1984, 1992, 1993, 1994) (§2.3), based on a Lagrangian description of surface waves. In §3 we show how to include damping as a weak perturbation via the addition of damping terms

to the amplitude equation (§3.1), and we relate the damping terms to the energy dissipation (§3.2), using an exact energy dissipation formula, which we derive directly from the Navier–Stokes equation in Appendix C. We only consider viscous bulk damping, neglecting dissipation from surface contamination and boundaries. It is essential when deriving the nonlinear terms in the amplitude equation that damping is correctly treated, and therefore we shall discuss this in detail. In §4, we compare our results, derived from the Hamiltonian approach (detailed in Appendices A, D, E), with results previously obtained.

The final results for the amplitude equation for travelling waves, is of the form (Milner 1991)

$$0 = \dot{a}_j - \mathrm{i}c_0 f a^*_{-j} + \gamma^{(0)} a_j + \mathscr{G}_j a_j + \sum_l (\gamma^{(1)}_{jl} - \mathrm{i}T^{(1)}_{jl})|a_l|^2 a_j + \sum_l (\gamma^{(2)}_{jl} - \mathrm{i}T^{(2)}_{jl}) a_l a_{-l} a^*_{-j}, \quad (1)$$

where

$$\mathscr{G}_j = c_1 \hat{\boldsymbol{k}}_j \cdot \nabla_\perp + \mathrm{i}c_2 (\hat{\boldsymbol{k}}_j \cdot \nabla_\perp)^2 - \mathrm{i}c_3 \nabla^2_\perp. \quad (2)$$

Here * denotes complex conjugation, $-j$ refers to the wave vector $-\boldsymbol{k}_j$, and $\nabla_\perp = (\partial_x, \partial_y)$, c_0 represents the coupling to the drive, $\gamma^{(0)}$ and $\gamma^{(i)}_{jk}$ are damping coefficients, and $T^{(i)}_{jk}$ are nonlinear detuning coefficients. In (2), $\hat{\boldsymbol{k}}_j = \boldsymbol{k}_j/k$, c_1 represents the group velocity, and c_2, c_3 are dispersion coefficients (see §2.1). The values of the nonlinear coefficients are given in §4, and compared to the results reported by others.

The reduction of the amplitude equation for standing waves (§5.1, Appendix F) is enforced by taking the solutions of a truncated linear stability analysis (Riecke 1990; Milner 1991) as the neutral solutions in a multiple-scales expansion of the amplitude equations for travelling waves. We also give a direct derivation of the standing wave equation from Miles' theory, in terms of a real amplitude and phase (as in Ezerskii *et al.* 1986). The final form of the amplitude equation for standing waves is

$$0 = \dot{A}_j + (\gamma^{(0)} - c_0 f) A_j - \frac{1}{2\gamma^{(0)}} \mathscr{G}^2_j A_j + \sum_{l=1}^{2N} \gamma_{jl} |A_l|^2 A_j, \quad (3)$$

where $\mathscr{G}_j$ is given by the first term in (2), N is the number of standing waves ($N = 1$ for rolls, $N = 2$ for squares, etc.), and

$$\gamma_{jl} = \tfrac{1}{2}[\gamma^{(1)}_{jl} + \gamma^{(2)}_{jl} + \gamma^{(1)}_{-jl} + \gamma^{(2)}_{-jl}]. \quad (4)$$

It is obvious that a correct theory of pattern selection in the Faraday experiment rests on correctly derived amplitude equations. One might for instance ask to what extent we have theoretical support for the selection of square patterns or quasi-crystalline patterns via the potential theory for standing waves. In §5.2, we examine the formation of regular patterns. We find that the square pattern has a noticeable subcriticality. Among the supercritical patterns examined, we find that the eightfold quasi-crystalline pattern, observed by Christiansen *et al.* (1992) and by Bosch (1995), is the preferred pattern. Although the theory may be of limited applicability to the existing experimental results (Christiansen *et al.* 1995; Bosch 1995), we shall discuss experimental observations in the light of the theory (§6). In particular, we consider finite-size effects (§6.2), and effects associated with gravity and viscosity (§6.3).

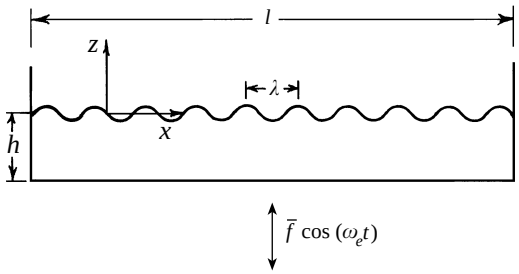

FIGURE 1. The hydrodynamical system (see text).

2. The nonlinear theory at zero viscosity

Below, we shall review the main formulations of a nonlinear theory describing the Faraday instability at low viscosity and in large-aspect-ratio systems. The first (§2.1), the Hamiltonian formulation, is due to Milner (1991) (Zakharov 1968; Zakharov *et al.* 1971, 1992; Ezerskii *et al.* 1986). The fundamental hydrodynamical equations are introduced, and we outline the basis for deriving the amplitude equation for travelling waves in the absence of damping, using a multiple-scales or Newell–Whitehead expansion of the hydrodynamical equations. The detailed derivation is found in Appendix A. The Hamiltonian theory has been considered in an alternative form by Zhang & Viñals (1996, 1997), and we outline this theory in §2.2. A second description, based on a Lagrangian formulation, is due to Miles (1976, 1977, 1984, 1992, 1993, 1994), and we shall consider this formulation in §2.3. The Lagrangian description, also used by Umeki & Kambe (1989), Kambe & Umeki (1990) and Umeki (1991, 1996) may be used to derive equations of motions for the real and imaginary parts of the amplitudes. Our goal is to review the theories in such a way that it is possible to make a quantitative comparison between the various descriptions, and a quantitative comparison between our results and the results obtained by others. The detailed comparison of results is relegated to §4. Here we shall mainly be concerned with the basic structure of the Hamiltonian and the Lagrangian description. In §3 we show how to include damping in the description (Hamiltonian as well as Lagrangian).

We consider a fluid characterized by its density, ρ, its dynamic viscosity, $\eta = \rho\nu$, and its surface tension, σ. The fluid resides in a container with a characteristic lateral extension (diameter or width) l, and a fluid depth h (figure 1). Both length scales are assumed to be large – in the formulations below, both scales are taken to be infinite. The container is subject to vertical oscillations, $\bar{f}\cos(\omega_e t)$, with amplitude $\bar{f}$ and frequency ω_e, and the phenomenon under study is the generation of waves on the surface of the fluid enforced by the oscillation of the container. We shall analyse the problem in a frame that is at rest relative to the container and with axes chosen as in figure 1.

2.1. *The Hamiltonian approach*

The starting point for the Hamiltonian description (Milner 1991) is the hydrodynamical equations for an ideal and incompressible fluid subject to vertical oscillations. In the rest frame of the container, the equation of motion and the incompressibility condition take the form

$$\dot{\phi} = -\tfrac{1}{2}(\nabla\phi)^2 - \frac{p}{\rho} + zf\cos(\omega_e t), \tag{5}$$

$$\nabla^2\phi = 0. \tag{6}$$

Here $\phi(x, y, z)$ is the velocity potential, defined such that $\boldsymbol{v} = \nabla\phi$, $\boldsymbol{v}$ being the velocity. (In general, the fields we consider depend on time t. However, for simplicity, we usually specify only the spatial variables.) p is the pressure field and $f = \omega_e^2 \bar{f}$ is the forcing amplitude of the effective gravitational force experienced by the fluid when being vertically vibrated at frequency ω_e in the capillary wave limit $kl_c \gg 1$.

At the free surface, $z = \zeta(x, y)$, the equation of motion reads

$$\dot{\phi}_\zeta = -\tfrac{1}{2}(\nabla\phi_\zeta)^2 - \frac{p_\zeta}{\rho} + \zeta f \cos(\omega_e t), \tag{7}$$

where subindex ζ refers to the surface $z = \zeta(\boldsymbol{x})$, using the vector notation $\boldsymbol{x} = (x, y)$. In addition, two boundary conditions are assigned at the free surface, namely the kinematic surface condition

$$\dot{\zeta} = -\nabla_\perp\zeta \cdot \nabla_\perp\phi_\zeta + \partial_z\phi_\zeta, \tag{8}$$

which is the statement of the existence of a free surface, and the Laplace formula

$$p_\zeta = -\sigma\nabla_\perp \cdot ([1 + (\nabla_\perp\zeta)^2]^{-1/2}\nabla_\perp\zeta), \tag{9}$$

which is the balance of forces along the normal of the free surface in the absence of viscous stresses.

Equations (6)–(9) are the set of fundamental equations used in order to derive an amplitude equation for travelling waves. To this end, Milner expands the velocity potential and its derivatives around $z = 0$, and we will do the same (see Appendix A, §A.1). The resulting linearized equation without forcing is

$$(\partial_t + \boldsymbol{L}_0)\begin{pmatrix} \zeta \\ \phi \end{pmatrix} = 0; \quad \boldsymbol{L}_0 = \begin{pmatrix} 0 & -\partial_z \\ -(\sigma/\rho)\nabla_\perp^2 & 0 \end{pmatrix}. \tag{10}$$

For a perturbation of the form $(\zeta_{\boldsymbol{k}}, \phi_{\boldsymbol{k}}\mathrm{e}^{kz})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ (the z-dependence is enforced by the incompressibility condition, (6), and the boundary condition $h \to \infty$), $\boldsymbol{L}_0$ has the matrix form

$$\bar{\boldsymbol{L}}_0 = \begin{pmatrix} 0 & -k \\ (\sigma/\rho)k^2 & 0 \end{pmatrix}. \tag{11}$$

The eigenvalues of $\bar{\boldsymbol{L}}_0$ are $\pm\mathrm{i}\omega$, where ω is related to k via the dispersion relation

$$\omega^2 = \frac{\sigma}{\rho}k^3. \tag{12}$$

A right eigenvector associated with the eigenvalue $\mathrm{i}\omega$ is $\boldsymbol{\Psi}_r = (1, -\mathrm{i}\omega/k)$; hence, the marginally growing or neutral solutions for ζ and ϕ are

$$\zeta_0 = \sum_j a_j \exp(\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega t)) + \text{c.c.}, \tag{13}$$

$$\phi_0 = \frac{-\mathrm{i}\omega}{k}\mathrm{e}^{kz}\sum_j a_j \exp(\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega t)) + \text{c.c.} \tag{14}$$

(c.c. denoting the complex conjugate), for some set, $\{\boldsymbol{k}_j\}$, of wave vectors of equal magnitude, $|\boldsymbol{k}_j| = k$, related to the frequency via the dispersion relation, (12). From the linear stability analysis with forcing by Benjamin & Ursell (1954), the Faraday instability is known to be parametrically driven, i.e. $\omega = \omega_e/2$.

The selection of a limited set of wave vectors and the saturation of the amplitudes are not described by linear theory which, at a given frequency, allows for a whole

range of equally unstable wave vectors lying on a circle in $\boldsymbol{k}$-space, and does not contain any statement about the magnitude of the amplitudes. To include these aspects in the description, Milner follows the philosophy of Newell & Whitehead (1969), and considers a wider class of solutions $a_j \to a_j(\boldsymbol{x}, t)$, where $a_j(\boldsymbol{x}, t)$ derives from the narrow band of unstable modes in $(\boldsymbol{k}, \omega)$-space, that may be excited just above the instability. The width of the unstable bands defines new length and time scales, and Milner chooses an expansion book-keeping parameter, here denoted χ, and scales the variables as $\boldsymbol{X} = \chi \boldsymbol{x}$, $T = \chi t$, $(\zeta, \phi) \to \chi(\zeta_0, \phi_0) + \chi^2(\zeta_1, \phi_1) + \chi^3(\zeta_2, \phi_2)$, and $f \to \chi f$. He then performs an expansion of the nonlinear hydrodynamical equations for the surface displacement field ζ and the velocity potential ϕ, in order to find the $O(\chi^2)$ and $O(\chi^3)$ contributions (ζ_1, ϕ_1) and (ζ_2, ϕ_2) to the solutions (ζ, ϕ). The equations obtained for these contributions have solvability conditions that add terms to the amplitude equation for a_j. The Newell–Whitehead expansion is carried out in detail in Appendix A.

The general amplitude equation obtained for travelling waves, (1), has a form that can be written down without reference to the expansion of the hydrodynamics. It can be deduced from symmetry arguments (see e.g. Christiansen 1993), which require the equation to keep its form invariant under transformations of the amplitude corresponding to arbitrary rotations and translations in space, and restricted time translations, $t \to t + (2\pi/\omega_e) = t + (\pi/\omega)$. The latter symmetry is imposed by the coupling to the drive. Assuming ∂_t, ∇, and a_j to be of the same order of magnitude, χ, as required by Milner, the equation of motion for a_j to $O(\chi^3)$ in the absence of damping has the form

$$0 = \dot{a}_j - \mathrm{i}c_0 f a^*_{-j} + \mathscr{G}_j a_j - \mathrm{i}\sum_l T^{(1)}_{jl}|a_l|^2 a_j - \mathrm{i}\sum_l T^{(2)}_{jl} a_l a_{-l} a^*_{-j}, \tag{15}$$

where $\mathscr{G}_j$ is defined by (2). The coefficients $T^{(i)}_{jl}$ are the nonlinear detuning coefficients.

Milner notes that $a_j \propto \exp(\mathrm{i}(\Delta\boldsymbol{k} \cdot \boldsymbol{x} - \Delta\omega_j t))$ should be a solution to the linearized amplitude equation $0 = \dot{a}_j + \mathscr{G}_j a_j$ with $\Delta\omega_j = \Delta\omega_j(\Delta\boldsymbol{k}) \equiv \omega(\boldsymbol{k}_j + \Delta\boldsymbol{k}) - \omega(\boldsymbol{k})$ derived from the expansion of the dispersion relation, (12). From this requirement ($\mathscr{G}_j a_j = \mathrm{i}\Delta\omega_j a_j$), Milner obtains

$$c_1 = 3\omega/2k, \quad c_2 = 3\omega/8k^2, \quad c_3 = 3\omega/4k^2. \tag{16}$$

The coefficient in front of the driving term and the nonlinear detuning coefficients (derived in Appendix A) cannot be predicted on the basis of such simple arguments.

2.2. *The surface field approach*

As mentioned above, the velocity potential ϕ and its derivatives are here (and in Milner's work) expanded around $z = 0$. An alternative approach, used by Zhang & Viñals (1996, 1997), is to derive the nonlinear equations of motion, not at $z = 0$ but at the surface $z = \zeta(\boldsymbol{x})$, applying the so-called Dirichlet–Neumann operator $\hat{G}[\zeta]$, relating the surface value of the harmonic function ϕ to the value of its normal derivative (see e.g. Craig & Sulem 1993). The Dirichlet–Neumann operator is defined by

$$\hat{G}[\zeta](\phi_\zeta) \equiv (-\nabla_\perp \zeta, 1) \cdot \nabla \phi_\zeta, \tag{17}$$

which equals $\dot{\zeta}$ according to (8).

For the spatial and temporal derivatives, we have

$$\nabla_\perp(\phi_\zeta) = \nabla_\perp \phi_\zeta + \partial_z \phi_\zeta \nabla_\perp \zeta, \tag{18}$$

$$\partial_t(\phi_\zeta) = \dot{\phi}_\zeta + \partial_z \phi_\zeta \dot{\zeta}. \tag{19}$$

From these expressions and (8) for $\dot{\zeta}$, (7) can be recasted into an equation of motion for ϕ_ζ (Zhang & Viñals 1996, 1997 where $h = k\zeta$, and $\Phi = (k^2/\omega)\phi_\zeta$),

$$\partial_t(\phi_\zeta) = -\tfrac{1}{2}[\nabla_\perp(\phi_\zeta)]^2 + \tfrac{1}{2}[1 + (\nabla_\perp\zeta)^2](\partial_z\phi_\zeta)^2 - \frac{p_\zeta}{\rho} + \zeta f\cos(\omega_e t). \tag{20}$$

From (17)–(18), we get an expression for $\partial_z\phi_\zeta$,

$$\partial_z\phi_\zeta = [1 + (\nabla_\perp\zeta)^2]^{-1}[\hat{G}[\zeta](\phi_\zeta) + \nabla_\perp\zeta\cdot\nabla_\perp(\phi_\zeta)]. \tag{21}$$

This can be inserted into (20) to obtain the (viscous-free) equations of motion based on ζ and ϕ_ζ,

$$\dot{\zeta} = G[\zeta](\phi_\zeta), \tag{22}$$

$$\partial_t(\phi_\zeta) = -\tfrac{1}{2}[\nabla_\perp(\phi_\zeta)]^2 + \tfrac{1}{2}[1 + (\nabla_\perp\zeta)^2]^{-1}[\hat{G}[\zeta](\phi_\zeta) + \nabla_\perp\zeta\cdot\nabla_\perp(\phi_\zeta)]^2 - \frac{p_\zeta}{\rho} + \zeta f\cos(\omega_e t). \tag{23}$$

The expansion of the operator $\hat{G}[\zeta]$ to second order in ζ (Craig & Sulem 1993) is carried out in Appendix B. From this expansion, the expansion of the equations of motion to third order in the fields ζ and ϕ_ζ is straightforward (Zhang & Viñals 1996*a*, *b*). Zhang & Viñals derive higher-order corrections, here denoted $(\phi_\zeta)_n$, to the neutral solution. The expressions for $(\phi_\zeta)_n$ are directly related to ϕ_n,

$$(\phi_\zeta)_0(\boldsymbol{x}) = \phi_0(\boldsymbol{x}, 0), \tag{24}$$

$$(\phi_\zeta)_1(\boldsymbol{x}) = \phi_1(\boldsymbol{x}, 0) + \zeta_0(\boldsymbol{x})\partial_z\phi_0(\boldsymbol{x}, 0), \tag{25}$$

$$(\phi_\zeta)_2(\boldsymbol{x}) = \phi_2(\boldsymbol{x}, 0) + \zeta_0(\boldsymbol{x})\partial_z\phi_1(\boldsymbol{x}, 0) + \zeta_1(\boldsymbol{x})\partial_z\phi_0(\boldsymbol{x}, 0) + \tfrac{1}{2}\zeta_0(\boldsymbol{x})^2\partial_z^2\phi_0(\boldsymbol{x}, 0). \tag{26}$$

Our ideal fluid results for (ζ_0, ϕ_0) and (ζ_1, ϕ_1) (Appendix A) agree through (24)–(25) with the results by Zhang & Viñals (1997) for $(\zeta_0, (\phi_\zeta)_0)$ and $(\zeta_1, (\phi_\zeta)_1)$ (Zhang & Viñals do not compute $(\zeta_2, (\phi_\zeta)_2)$).

2.3. *The Lagrangian approach*

A natural starting point for Miles' description of surface waves is the observation (Miles 1977; Milder 1977) that the equations governing the hydrodynamics of surface waves can be derived from a variational principle that signals an underlying canonical formalism, allowing a change in variables from (ϕ_ζ, ζ) to $(\dot{\zeta}, \zeta)$. An essential new feature in Miles' description (also used by Umeki & Kambe (1989), Kambe & Umeki (1990), and Umeki (1991, 1996)) compared with Milner's is that it is performed in terms of a new set of variables.

To understand the nature of the Lagrangian theory let us briefly outline how this change of variables is carried out, by following Miles (1977), who defines a Lagrangian-like density, $\mathscr{L}$, as

$$\mathscr{L} = \rho\phi_\zeta\dot{\zeta} - \mathscr{H}, \quad \mathscr{H} = \mathscr{T} + \mathscr{V}, \tag{27}$$

where

$$\mathscr{T} = \int_{-h}^{\zeta} \mathrm{d}z\ \left[\frac{\rho}{2}\boldsymbol{v}^2\right], \quad \mathscr{V} = -\int_0^{\zeta} \mathrm{d}z\ [\rho f\cos(\omega_e t)z] + \sigma(g^{1/2} - 1) \tag{28}$$

(here $\zeta \equiv 0$ is taken as the zero point for the potential). We take $h \to \infty$, and g is the determinant of the metric tensor, which can be expressed in terms of ζ,

$$g = 1 + (\nabla_\perp\zeta)^2. \tag{29}$$

$\mathscr{T}$ and $\mathscr{V}$ are respectively the kinetic and the potential energy densities (per unit area), thus making $\mathscr{H}$ the density of mechanical energy, and

$$E = \int \mathrm{d}^2x\, \mathscr{H} = \int \mathrm{d}^2x\, [\mathscr{T} + \mathscr{V}] \tag{30}$$

the total mechanical energy. In Appendix D, we show, using Green's formula that

$$\int \mathrm{d}^2x\, \mathscr{T} = \frac{\rho}{2} \int \mathrm{d}^2x\, \phi_\zeta \dot{\zeta}, \tag{31}$$

i.e. $\mathscr{L}$ may be replaced by $\mathscr{T} - \mathscr{V}$.

Miles integrates the total Lagrangian, $L = \int \mathrm{d}^2x\, \mathscr{L}$, with respect to time to yield the action, $S = \int \mathrm{d}t\, L$, and he invokes that this action is stationary,

$$0 = \delta S = \delta \left(\int \mathrm{d}t\, L \right) = \delta \left(\int \mathrm{d}^2x\, \mathrm{d}t\, \mathscr{L} \right), \tag{32}$$

when the surface displacement and the velocity potentials are being varied independently, in such a way that the variations vanish at the rigid boundaries, $\partial\mathscr{B}$. It is an exercise in manipulations with functional derivatives to find that the basic hydrodynamical equations (7)–(8) can be derived from this variational principle, if the boundary condition $\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\nabla\phi = 0$ is imposed at $\partial\mathscr{B}$ in a finite container. As Miles notes, this signals an underlying canonical formalism (Hamiltonian formalism) for surface waves with ϕ_ζ and ζ playing the role of canonical variables (Zakharov 1968),

$$\rho\dot{\zeta} = \frac{\delta E}{\delta \phi_\zeta}, \quad \partial_t(\phi_\zeta) = -\frac{\delta E}{\delta \zeta}, \tag{33}$$

and the mechanical energy acting as the Hamiltonian. Equations (33) are exactly the (7)–(8) in §2.1. The Hamiltonian description of surface waves is thus a description in terms of ζ and ϕ_ζ. In that sense the analysis in §2.1 rests on a Hamiltonian description of surface waves.

In order to obtain the alternative, Lagrangian, description in terms of the variables $(\zeta, \dot{\zeta})$ Miles carries through a Legendre transformation, leading to $\mathscr{L} = \mathscr{L}[\zeta, \dot{\zeta}]$ via an inversion of the functional relation between $\dot{\zeta}$ and ϕ_ζ, as given by (33). Formally, the inverted expression can be written in terms of a suitable Green's function (Milder 1977),

$$\phi_\zeta(\boldsymbol{x}) = \int \mathrm{d}^2x'\, [G(\boldsymbol{x}, \boldsymbol{x}')\, \dot{\zeta}(\boldsymbol{x}')], \tag{34}$$

$$L = \frac{\rho}{2} \int \mathrm{d}^2x\, \mathrm{d}^2x'\, [\dot{\zeta}(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{x}') \dot{\zeta}(\boldsymbol{x}')] - V[\zeta], \tag{35}$$

which is not known in the general case. However, as Miles shows, it can be found in the case of a standing plane wave description. At this level it is possible for Miles to produce a description in terms of the variables $(\zeta, \dot{\zeta})$.

For laterally and horizontally infinite systems, Miles' standing plane wave description may be seen as an alternative approach to a multiple-scales description of surface waves, which proceeds in steps that resemble the Newell–Whitehead expansion very much and shares its goals, in that it seeks the derivation of an equation of motion for fields that we shall see is trivially related to the real and imaginary parts of the amplitude fields. However, the approach is more restrictive, since it already assumes that the relevant solutions for ζ are standing waves, i.e. $a_j = a_{-j}$, and it does not allow for slow spatial modulations of the amplitudes.

For an infinite system, the linear theory of surface waves allows Miles to take the lowest-order solutions for the surface displacement field to be standing waves

$$\zeta = \sum_{n=1}^{N} \eta_n(t)\psi_n(\boldsymbol{x};k_n), \quad \dot{\zeta} = \sum_{n=1}^{N} \dot{\eta}_n(t)\psi_n(\boldsymbol{x};k_n), \tag{36}$$

with

$$\psi_n(\boldsymbol{x};k_n) = (2/N)^{1/2}\cos(\boldsymbol{k}_n \cdot \boldsymbol{x}), \tag{37}$$

and

$$\eta_n(t) = \frac{2}{k}\epsilon^{1/2}(p_n\cos(\omega t) + q_n\sin(\omega t)), \tag{38}$$

for some set $n = 1, ..., N$ of modes forming a pattern ($N = 1$ for rolls, $N = 2$ for squares, etc.). Recall that $\epsilon = k\bar{f}$ for the viscous-free case discussed here. The wave vectors, $\boldsymbol{k}_n$, are all of equal magnitude, $|\boldsymbol{k}_n| = k$, and the frequency, ω, is half the driving frequency, $\omega = \omega_e/2$, and related via the dispersion relation, (12).

There is a simple relation between the position- and momentum-like variables, (q_n, p_n), introduced by Miles, and the amplitudes, a_j, introduced by Milner. In fact, apart from trivial constants, they translate into the real and imaginary parts of the amplitude field,

$$a_j = a_{-j} = a_n = \frac{1}{k}\left(\frac{\epsilon}{2N}\right)^{1/2}(p_n + \mathrm{i}q_n). \tag{39}$$

Like the amplitudes a_j in §2.1, the 'amplitudes' (q_n, p_n) obtained in linear theory are allowed to be slowly varying, though only in time. Miles takes $(q_n, p_n) \to (q_n(T), p_n(T))$, where $T = \epsilon\omega t$ is a dimensionless measure of the slow time scale, and derives an equation of motion for the slowly varying fields, which are now assumed to be of equal magnitude, $(q_n(T), p_n(T)) \to (q(T), p(T))$, and $a_n \to a$.

Miles derives only the $O(\epsilon)$ contributions to the solution for the surface displacement field. This is done by invoking the Euler–Lagrange equation implied by (32),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta L}{\delta\dot{\zeta}}\right) - \frac{\delta L}{\delta\zeta} = 0. \tag{40}$$

Inserting the results into his expression for his Lagrangian and averaging over the fast time, Miles obtains

$$\langle L\rangle = 4\sigma\epsilon^2[\tfrac{1}{2}(\dot{p}q - \dot{q}p) + H(p,q)], \tag{41}$$

where the time derivative is with respect to T, and $H(p,q)$ has the form

$$H(p,q) = \tfrac{1}{2}(p^2 - q^2) + \tfrac{1}{4}C(N)(p^2 + q^2)^2. \tag{42}$$

$H(p,q)$ is a new Hamiltonian, not to be confused with the mechanical energy, E. It enters in a novel canonical formalism that takes the slow temporal parts (q,p) of the surface displacement field as a conjugate pair, and, finally, allows Miles to generate equations of motion for the pair (q,p). This is effected by requiring $\langle L\rangle$ to be stationary with respect to independent variations of q and p, thus leading to the canonical equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \tag{43}$$

or, in the absence of damping,

$$\dot{p} = q - C(N)q(p^2 + q^2), \quad \dot{q} = p + C(N)p(p^2 + q^2). \tag{44}$$

Translating these results into expressions involving the amplitude a (via (39)) leads to the amplitude equation

$$0 = \dot{a} - \mathrm{i}\epsilon\omega a^* - \mathrm{i}2NC(N)\omega k^2|a|^2 a, \tag{45}$$

which has the expected form, considering the symmetries of the problem.

In comparison with (15), we have $c_0 f = \epsilon\omega$, or by using $\epsilon = k\bar{f}$ and $f = 4\omega^2\bar{f}$, we have

$$c_0 = k/4\omega, \tag{46}$$

in agreement with both Milner's result and ours (Appendix A). Comparing (45) with (15) also yields the relation

$$C(N) = \bar{T}(N) \equiv (2N\omega k^2)^{-1} \sum_l (T_{jl}^{(1)} + T_{jl}^{(2)}). \tag{47}$$

3. Damping

Here, we examine how damping is included in the Hamiltonian and the Lagrangian descriptions of low-viscosity capillary surface waves. We shall only take into account the bulk damping, and boundaries are assumed to be at infinity ($kl = \infty$, $kh = \infty$). Moreover, we only consider damping to $O(\nu)$.

We have already derived an equation for the amplitudes (or its real and imaginary parts) in the absence of damping, i.e. for an ideal fluid. The assumption, applied in the Hamiltonian as well as the Lagrangian description, is that weak damping can be accounted for by adding damping terms to the equation for the ideal fluid amplitudes, and the damping coefficients can be calculated on the basis of ideal fluid solutions by including dissipation on a 'slow' scale. Previously, it has not been clear to what extent this assumption is valid. Below, we show that the assumption is correct to $O(\nu)$.

In §3.1 we outline how to incorporate the viscosity in the perturbation theory, leading to the additional terms in the amplitude equation. To calculate these terms, we apply an energy-dissipation formula, which we show in §3.2 to be exact to $O(\nu)$.

3.1. *Nonlinear theory with viscosity*

The presence of viscosity gives rise to additional terms in the equations of motion. To lowest order, these terms are linear in viscosity, and here we only consider corrections to that order. In particular, the dispersion relation, (12), has no correction to first order in viscosity (see e.g. Kumar 1996). The extra terms in the equations of motion lead to corrections, $\mu(\zeta_0^{(\nu)}, \phi_0^{(\nu)})$, $\mu(\zeta_1^{(\nu)}, \phi_1^{(\nu)})$, $\mu(\zeta_2^{(\nu)}, \phi_2^{(\nu)})$, to the solutions (ζ_0, ϕ_0), (ζ_1, ϕ_1), (ζ_2, ϕ_2), where we have introduced the dimensionless parameter $\mu = \nu k^2/\omega$, which is assumed small (low viscosity). For example,

$$(\partial_t + \boldsymbol{L}_0)\begin{pmatrix} \zeta_0^{(\nu)} \\ \phi_0^{(\nu)} \end{pmatrix} = -(\partial_{\mu t} + \boldsymbol{L}_0^{(\nu)})\begin{pmatrix} \zeta_0 \\ \phi_0 \end{pmatrix};\ \boldsymbol{L}_0^{(\nu)} = \begin{pmatrix} -2\dfrac{\omega}{k^2}\nabla_\perp^2 & 0 \\ -\dfrac{f_c}{\mu}\cos(\omega_e t) & -2\dfrac{\omega}{k^2}\nabla_\perp^2 \end{pmatrix}, \tag{48}$$

where the additional time scale μt has been introduced: $\partial t \to \partial t + \mu\partial_{\mu t}$. Also the additional length scale $\mu\epsilon^{1/2}\boldsymbol{x}$ enters, changing the natural scaling length in the Newell–Whitehead expansion from $\epsilon\boldsymbol{x}$ to $\epsilon^{1/2}\boldsymbol{x}$ (see §5 and Appendix F).

The solvability condition from (48) yields additional terms to the amplitude equation,

$$\mu\partial_{\mu t} a_j = \mathrm{i}c_0 f_c a_{-j}^* - \gamma^{(0)} a_j, \tag{49}$$

with $c_0 = k/4\omega$ and $\gamma^{(0)} = 2\nu k^2$. Note that the value of f_c is determined by the condition that $\partial_{\mu t} a_j = 0$. Hence, we have $f_c = \gamma_0/c_0 = 8\nu\omega k$. At this forcing, $a_j = \mathrm{i}a^*_{-j}$, thus the neutral solutions at non-zero viscosity are standing waves.

The solution to (48) is

$$\zeta_0^{(v)} = -\frac{1}{2}\sum_j a_j \exp(\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - 3\omega t)) + \text{c.c.}, \tag{50}$$

$$\phi_0^{(v)} = \frac{-\mathrm{i}\omega}{k}\mathrm{e}^{kz}\sum_j a_j[2\mathrm{i}\exp(\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega t)) - \tfrac{3}{2}\exp(\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - 3\omega t))] + \text{c.c.} \tag{51}$$

This correction to the solution is also found by Zhang & Viñals (1996, 1997).

The presence of linear viscous terms in the equations of motion, and the presence of higher harmonics in the viscous corrections to the neutral solution give rise to natural concerns regarding the energy dissipation formalism that Milner uses when introducing viscosity. In the description that follows, we shall address such concerns.

In Milner's Hamiltonian description viscosity enters by adding to the amplitude equation, (15), the lowest-order relaxational terms allowed by symmetry,

$$\dot{a}_j = \ldots - \gamma^{(0)} a_j - \sum_l \gamma^{(1)}_{jl}|a_l|^2 a_j - \sum_l \gamma^{(2)}_{jl} a_l a_{-l} a^*_{-j}. \tag{52}$$

Does this approach give the correct linear viscous terms? The answer is yes. Equation (52) introduces dissipative terms $\dot{\zeta}^{(dis)}$ in the equation for ζ, as well as dissipative terms $\dot{\phi}^{(dis)}$ in the equation for ϕ (and not only in the equation for ϕ, cf. Zhang & Viñals 1997). A simple calculation shows that the terms $\dot{\zeta}^{(dis)}$ and $\dot{\phi}^{(dis)}$ contribute equal amounts to the linear dissipation, giving rise to equal diagonal elements $(-2(\omega/k^2)\nabla^2_\perp, -2(\omega/k^2)\nabla^2_\perp)$ in $\boldsymbol{L}_0^{(v)}$ (equation (48)).

Also in the Lagrangian approach, dissipative terms enter both equations of motion. Miles introduces a dissipation function, here denoted $d(p, q)$, from which the damping terms in the equations for $\dot{p}$ and $\dot{q}$ are derived. Symmetry arguments (Miles 1993) suggest

$$d(p, q) = \tfrac{1}{2}\alpha(p^2 + q^2) + \tfrac{1}{4}\gamma(N)(p^2 + q^2)^2, \tag{53}$$

from which Miles derives the damping terms

$$\dot{p} = -\frac{\partial d}{\partial p} + \ldots = -\alpha p - \gamma(N)(p^2 + q^2)p + \ldots, \tag{54}$$

$$\dot{q} = -\frac{\partial d}{\partial q} + \ldots = -\alpha q - \gamma(N)(p^2 + q^2)q + \ldots. \tag{55}$$

In terms of the amplitude field a, (54)–(55) translate into

$$\dot{a} = \ldots - \alpha\epsilon\omega a - 2N\gamma(N)\omega k^2|a|^2 a + \ldots. \tag{56}$$

In comparison with (52), we have the relations

$$\alpha\epsilon\omega = \gamma^{(0)}, \quad \frac{\omega}{\nu k^2}\gamma(N) = \bar{\gamma}(N) \equiv (2N\nu k^4)^{-1}\sum_l(\gamma^{(1)}_{jl} + \gamma^{(2)}_{jl}). \tag{57}$$

Another concern that naturally arises is that certain contributions to the nonlinear damping, e.g. terms arising from the linear viscous terms in the fluid equations, are absent in Milner's energy dissipation approach. In the next section (§3.2) we shall show that this approach to determining the nonlinear damping terms is not only 'phenomenologically' correct, but also analytically correct to first order in viscosity.

Thus all $O(v)$ contributions to the nonlinear damping can be included. In contrast, Zhang & Viñals only consider contributions to the nonlinear damping from the linear viscous terms, but they completely neglect all nonlinear viscous terms $\boldsymbol{N}_i^{(v)}$ in the equations for $(\zeta_i^{(v)}, \phi_i^{(v)})$ $(i = 1, 2)$,

$$(\partial_t + \boldsymbol{L}_0) \begin{pmatrix} \zeta_i^{(v)} \\ \phi_i^{(v)} \end{pmatrix} = -\boldsymbol{L}_0^{(v)} \begin{pmatrix} \zeta_i \\ \phi_i \end{pmatrix} - \cdots - \boldsymbol{N}_i^{(v)} \tag{58}$$

(the dots refer to the other linear terms). For example, Zhang & Viñals neglect the nonlinear viscous terms arising from the pressure p_ζ that differs from the ideal fluid result $p_\zeta^{(ideal)}$, given by (9). The difference between the viscous and ideal fluid surface pressure $p_\zeta - p_\zeta^{(ideal)}$ has nonlinear terms of both second order (of the form $\nabla_\perp \zeta \cdot \nabla_\perp \partial_z \phi$) and third order (for example of the form $[(\nabla_\perp \zeta \cdot \nabla_\perp) \nabla_\perp \phi] \cdot \nabla_\perp \zeta$).

3.2. *Energy dissipation*

The issue is to determine the damping coefficients $\gamma^{(0)}$, $\gamma_{jl}^{(1)}$, and $\gamma_{jl}^{(2)}$, introduced by (52). The same method is followed in both the Hamiltonian and Lagrangian description. The starting point is the following exact formula for the loss of mechanical energy $\dot{E}$ due to viscous stresses, and valid for incompressible fluids:

$$\dot{E} = D + \int \mathrm{d}^2x \, [\tfrac{1}{2} \rho f \omega_e \sin(\omega_e t) \zeta^2], \tag{59}$$

where

$$D \equiv -\frac{\eta}{2} \int \mathrm{d}^3x \, (\partial_i v_j + \partial_j v_i)^2 \tag{60}$$

($\int \mathrm{d}^3x = \int \mathrm{d}^2x \int_{-\infty}^{\zeta} \mathrm{d}z$). We emphasize that the forcing integral in (59) was missing in Milner (1991). Since the analytical correctness of (59) is crucial for calculating the nonlinear damping, we have derived this formula in Appendix C. We note that the energy is conserved on a fast scale.

Both the mechanical energy, E, and the energy dissipation function, D, are now calculated on the basis of the results obtained from the multiple-scales theory valid for the ideal fluid. The relevant terms in the expansions of E and D have the form

$$\langle E \rangle = \sum_j H^{(0)} |a_j|^2 + \sum_{j,l} [H_{jl}^{(1)} |a_l|^2 |a_j|^2 + H_{jl}^{(2)} a_l a_{-l} a_j^* a_{-j}^*], \tag{61}$$

and

$$\langle D \rangle = -\sum_j D^{(0)} |a_j|^2 - \sum_{j,l} [D_{jl}^{(1)} |a_l|^2 |a_j|^2 + D_{jl}^{(2)} a_l a_{-l} a_j^* a_{-j}^*]. \tag{62}$$

The fact that $f_c \neq 0$ for $v > 0$ gives rise to speculations on which forcing contributions that appear in the amplitude equation. As shown in Appendix D, §D.4, and Appendix E, §E.3, $\langle \dot{E} \rangle$ will contain terms of the form $\mu\omega\rho f [a_j a_{-j} + a_j^* a_{-j}^*]$ and $\mathrm{i}\omega\rho f [a_j a_{-j} - a_j^* a_{-j}^*]$. These contributions will however be exactly balanced by equal terms on the right-hand side of (59), and will not add new terms to the amplitude equation. The higher-harmonic corrections of order $\mu \exp(\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - 3\omega t))$ to the surface field solution give at first sight corrections to the forcing integral in (59); however, because of the sinusoidal function, the contribution to the average dissipation turns out to be identically zero. There will be terms to the dissipation of the form $f|a_l|^2 a_j a_{-j}$ (and c.c.), giving rise to terms in the amplitude equation of the form $f a_l^* a_{-l}^* a_j$ and

$f|a_l|^2 a^*_{-j}$, which may play a role in the detuning of wave patterns. We do not consider such terms here.

Based on the amplitude equation, Milner gives results for the H and the D, which are surprisingly simple in the case of the H:

$$H^{(0)} = 2\sigma k^2, \tag{63}$$

$$H^{(i)}_{jl} = -(\sigma k^2/\omega) T^{(i)}_{jl}. \tag{64}$$

In Appendix D we show that while the first equation is correct, the second is not. The D are derived from an expansion of the right-hand side of (60), and calculated from the multiple-scales theory (Appendix E). By taking the time derivative of the energy E, and expressing the $\dot{a}_j$ and $\dot{a}^*_j$, thus generated via the damping terms (52) in the amplitude equation, one obtains the identities

$$\gamma^{(0)} = \frac{D^{(0)}}{2H^{(0)}}, \tag{65}$$

$$\gamma^{(i)}_{jl} = \frac{1}{2H^{(0)}} D^{(i)}_{jl} - \frac{2\gamma^{(0)}}{H^{(0)}} H^{(i)}_{jl}. \tag{66}$$

The $O(v)$ corrections $(\zeta^{(v)}_i, \phi^{(v)}_i)$ $(i = 1, 2)$ to the solution give $O(v)$ corrections to the mechanical energy, and $O(v^2)$ corrections to the energy dissipation function. By (65)–(66), the resulting contributions to $\gamma^{(0)}$ and $\gamma^{(i)}_{jl}$ are of order v^2. This shows how useful the energy dissipation approach is to first order in viscosity. The nonlinear damping of $O(v)$ is derived directly from the Navier–Stokes equations and the ideal fluid results. We note that the contributions to the nonlinear damping from the last term on the right-hand side of (66) is related to the linear viscous terms in the equations of motion.

The damping coefficients $\gamma^{(0)}$, $\gamma^{(i)}_{jl}$, can also be derived without explicitly calculating the average energy $\langle E \rangle$. Alternatively, one may use the Hamiltonian equations, (33), and the fact that

$$\dot{E} = \int \mathrm{d}^2 x \left[\frac{\delta E}{\delta \phi_\zeta} \partial_t(\phi_\zeta) + \frac{\delta E}{\delta \zeta} \dot{\zeta} \right]. \tag{67}$$

It follows that

$$\dot{E} = \rho \int \mathrm{d}^2 x \, [\dot{\zeta}^{(ham)} \dot{\phi}^{(dis)}_\zeta - \dot{\phi}^{(ham)}_\zeta \dot{\zeta}^{(dis)}], \tag{68}$$

where *ham* refers to the Hamiltonian terms, (15), and *dis* refers to the dissipative terms, (52). As already noted in §3.1, the terms $\dot{\zeta}^{(ham)} \dot{\phi}^{(dis)}_\zeta$ and $-\dot{\phi}^{(ham)}_\zeta \dot{\zeta}^{(dis)}$ contribute equally (both with $4v\sigma k^4$) to the linear dissipation.

Miles also considers the formula for the loss of mechanical energy due to viscous stresses, although in a slightly different form. He also calculates the total mechanical energy E, as well as the energy dissipation function D, which in his notation are of the form

$$\langle E \rangle = \sigma [2\epsilon(p^2 + q^2) + \hat{C}(N)\epsilon^2 (p^2 + q^2)^2], \tag{69}$$

and

$$\langle D \rangle = -4\sigma\omega [\delta\epsilon(p^2 + q^2) + \Gamma(N)\epsilon^2 (p^2 + q^2)^2]. \tag{70}$$

From (54)–(55) Miles obtains the identities

$$\alpha = \delta/\epsilon, \quad \gamma(N) = \Gamma(N) - \delta \hat{C}(N). \tag{71}$$

By (57) and (71) we have $\omega\delta = \gamma^{(0)}$. Applying the relation (39) between a and (p, q), a comparison between (61)–(62) and (69)–(70) reproduces to second order the results (63) and (65). To fourth order, we have the relations

$$\hat{C}(N) = \bar{H}(N) \equiv (2N\sigma k^4)^{-1} \sum_l (H_{jl}^{(1)} + H_{jl}^{(2)}), \tag{72}$$

and

$$\frac{4\omega}{\nu k^2} \Gamma(N) = \bar{D}(N) \equiv (2N\nu\sigma k^6)^{-1} \sum_l (D_{jl}^{(1)} + D_{jl}^{(2)}). \tag{73}$$

From relations (47) and (72), it follows that Milner's equation (64) upon reduction to standing waves reduces to $\bar{H}(N) = -\bar{T}(N)$, or in Miles' Lagrangian formulation $\hat{C}(N) = -C(N)$. This is incorrect, as pointed out by Miles (1993).

4. The nonlinear coefficients

As we saw in §2, both the Hamiltonian and the Lagrangian descriptions of surface waves ultimately rely on a multiple-scales expansion of the basic equations. Thus, both approaches yield amplitude equations for travelling waves. In Milner's formulation, the form is given by (15), supplemented with the damping terms, (52). In Miles' formulation, the form is given by (44), with the damping terms (54)–(55).

Here, we discuss to what extent the coefficients from the two descriptions are in agreement, and to what extent they agree with our results. Our calculations, presented in the Appendices, are detailed to make it possible for others to check them readily. The calculations may be seen as an independent check of the Newell–Whitehead expansion performed by Milner. Thus the Appendices contain results obtained within the framework of a Hamiltonian description of surface waves, as presented in §§2.1 and 2.2. We have, however, also sought to make the calculations comparable with the calculations performed by Miles. We have, for instance, derived expressions for the Lagrangian in terms of amplitude variables. Appendix A contains the essentials of the Newell–Whitehead expansion, Appendix D contains an expansion of the mechanical energy and the Lagrangian in terms of the amplitudes, while Appendix E contains an expansion of the energy-dissipation function D.

The present section is organized as follows. In §4.1 we consider the nonlinear detuning coefficients $T_{jl}^{(i)}$ in the amplitude equation in the absence of damping. We compare our results with those reported by Milner (1991) and Miles (1993). In §4.2 we give the damping coefficients $\gamma_{jl}^{(i)}$ (and $\gamma^{(0)}$) entering the damping terms in the amplitude equation. Again, we compare our results with those obtained in the aforementioned papers by Milner and Miles.

Before we consider the amplitude equations we make three comments. First, there seems to be agreement between Milner and Miles that the $O(\epsilon)$ stationary surface displacement field, ζ_1, will contain no 'zero-point' contribution $\sum_j |a_j(x,t)|^2$ (a lifting of the surface on a slow scale). As shown in the Appendices, the removal of the 'zero-point' contribution in the expression for ζ_1 gives rise to a breakdown of the usual combinatorial rule that the nonlinear coefficients at zero angle between wave vectors equals half the limit value obtained for the angle converging to zero. Since this has consequences for the pattern formation, we have carefully followed the terms resulting from the removal of the 'zero-point' contribution. We emphasize that this breakdown arises solely because we neglect gravitation. We shall elaborate on this further below.

Secondly, both Milner and Miles exclude the neutral contributions at $O(\epsilon)$ and $O(\epsilon^{3/2})$ to the solutions for ζ and ϕ. However, one cannot on mathematical grounds exclude higher-order resonant corrections b_j, c_j, to the amplitude field, a_j, itself (see e.g. Manneville 1990). For completeness, we have retained such corrections in our Newell–Whitehead expansion in Appendix A. However, we do not include them in our calculation of the mechanical energy E (Appendix D) and the energy dissipation function D (Appendix E), since it turns out that the extra coupling terms involving the higher-order resonant corrections exactly cancel each other. Thus, the higher-order resonant corrections do not enter the amplitude equations (with or without damping terms).

Thirdly, there is a difference in the scaling of the slow spatial and temporal variables applied by Milner and Miles. In the absence of damping the shape of the resonance tongues from linear theory implies that the bandwidth in δk grows $\sim f$. Moreover, the group velocity is finite, $\Delta\omega \sim \Delta k$. Using $\epsilon = k\bar{f}$, we thus have the natural scaling $X = \epsilon x$ and $T = \epsilon t$. (In the presence of damping, the bottom of the resonance tongues are rounded, $\delta k \sim (f - f_c)^{1/2}$, and the natural scaling would be $X = \epsilon^{1/2}x$, $T = \epsilon^{1/2}t$, where $\epsilon = k(\bar{f} - \bar{f}_c)$.)

As already noted, Milner's expansion of the hydrodynamical equations, leading to the amplitude equation displayed in (15), follows from the scaling $\boldsymbol{X} = \chi\boldsymbol{x}$, and $T = \chi t$, with no slow variable in the z-direction, where $f \to \chi f$, and $(\zeta, \phi) \to \chi(\zeta_0, \phi_0)$, at lowest order. The amplitude equation derived this way gets contributions from different orders of the book-keeping parameter χ. Implicitly, it contains two amplitude equations at different orders, $O(\chi^2)$ and $O(\chi^3)$. These two equations can be thought of as equations of motion for the amplitudes on two slow time scales, namely the time scale characterizing the propagation of wave packets travelling with the group velocity, and the time scale characterizing the dispersion of wave packets. Milner adds to the amplitude equation for the former time scale, at which the forcing also enters, the next-order solvability condition so as to include nonlinearities. These come in at the order where dispersion terms enter the description, due to the basic scaling.

For the scaling of the amplitudes, it turns out to be technically easier (and in accordance with the standard approach to pattern formation) to bring in nonlinearities at the scale characterizing propagation of wave packets, i.e. to choose the scaling $\boldsymbol{X} = \epsilon x$, $T = \epsilon t$, and $(\zeta, \phi) \to \epsilon^{1/2}(\zeta_0, \phi_0) + \ldots$, where $\epsilon = k\bar{f}$. This scaling is exactly the one underlying the analysis by Miles and the Newell–Whitehead expansion performed in Appendix A. It again leads to the amplitude equation, (15), however with no dispersion terms in $\mathscr{G}_j$,

$$\mathscr{G}_j = c_1 \hat{\boldsymbol{k}}_j \cdot \nabla_\perp, \tag{74}$$

where $c_1 = 3\omega/2k$. The choice of scaling (Milner's or Miles') does not influence the nonlinear cubic terms.

4.1. *Nonlinear detuning*

We now discuss and compare the reported coefficient for the ideal fluid amplitude equations, (15). The values for c_i, (16) and (46), are all agreed upon (c_0 and c_1 are also derived in Appendix A), so we focus on the nonlinear detuning terms $T_{jl}^{(i)}$. From the Newell–Whitehead expansion carried out in Appendix A, we find

$$\begin{aligned} T_{jl}^{(1)} &= T^{(1)}(\theta) \\ &= (1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)[d - \tfrac{1}{4}(3 + 2c)b - 2 - 4c + c^2]\omega k^2, \end{aligned} \tag{75}$$

$$T_{jl}^{(2)} = T^{(2)}(\theta) = \tfrac{1}{2}(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)(3 + c^2)\omega k^2, \tag{76}$$

where $\theta = \theta_{jk}$ is the angle between wave vectors $\boldsymbol{k}_j$ and $\boldsymbol{k}_k$, and

$$c = c_{jk} = \hat{\boldsymbol{k}}_j \cdot \hat{\boldsymbol{k}}_k = \cos(\theta_{jk}), \tag{77}$$

$$c_+ = \frac{|\boldsymbol{k}_j + \boldsymbol{k}_l|}{2k} = \left(\frac{1+c}{2}\right)^{1/2}, \tag{78}$$

and where

$$d = \frac{(3-c)c_+ - 2(1+c)}{(1+c)c_+ - 1}, \tag{79}$$

$$b = \frac{1 - 5c - 2c^2}{(1+c)c_+ - 1} \tag{80}$$

(Milner 1991), is the solution to the linear equations

$$-d + c_+ b = -2(1+c), \tag{81}$$

$$(1+c)d - b = 3 - c. \tag{82}$$

To avoid double counting at $j = k$ and $j = -k$, a pair of Kronecker δ-functions $\delta_+ = \delta_{jk}$ and $\delta_- = \delta_{-jk}$ is introduced (see Appendix A). The inclusion of these δ-functions leads to the multiplication of the nonlinear terms by a factor $(1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)$. The δ-function stemming from the restricted sum in ζ_1 (no 'zero-point' term) gives no contribution to $T^{(i)}_{jk}$. Thus,

$$T^{(i)}(0) = \tfrac{1}{2} \lim_{\theta \to 0} T^{(i)}(\theta), \quad T^{(i)}(\pi) = \tfrac{1}{2} \lim_{\theta \to \pi} T^{(i)}(\theta). \tag{83}$$

Our results for $T^{(i)}_{jk}$, (75)–(76), are identical to those reported by Milner, aside from the factors of $(1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)$ which are missing in Milner's paper. In figure 2(a) we show $T^{(1)}(\theta)$, $T^{(2)}(\theta)$, and the symmetric function

$$T(\theta) \equiv \tfrac{1}{2}[T^{(1)}(\theta) + T^{(2)}(\theta) + T^{(1)}(\pi - \theta) + T^{(2)}(\pi - \theta)]. \tag{84}$$

At $\cos(\theta) = 2^{1/3} - 1$ the nonlinearities produce waves resonant with the drive frequency. This gives spurious divergences, since these waves are excluded in the calculation (Milner 1991).

For standing waves, we have (cf. (47)),

$$\frac{1}{N} \sum_{l=1}^{N} T(\theta_{jl}) = \omega k^2 \bar{T}(N). \tag{85}$$

For rolls, squares, hexagons and the eightfold quasi-crystal, corresponding to $N = 1$, $N = 2$, $N = 3$, and $N = 4$, we find $\bar{T}(1) = 17/8 = 2.125$, $\bar{T}(2) = (33 - 2\sqrt{2})/16 \simeq 1.89$, $\bar{T}(3) = (485 + 64\sqrt{3})/264 \simeq 2.26$, and $\bar{T}(4) \simeq 2.36$. These values are in agreement with Miles' results for rolls and squares, recalling that $\bar{T}(N)$ equals $C(N)$ in Miles' formulation. The value 11.79 of $4\bar{T}(2)$ reported by Milner is wrong by $2\bar{T}(1) = 17/4$ from the correct value, $4\bar{T}(2) \simeq 7.54$, because the δ-functions at $j = k$ and $j = -k$ were not taken into account.

It appears to us that Miles in his derivation of the average Lagrangian has excluded the $O(\epsilon^{3/2})$ resonant contributions to the solution for the surface displacement field. One cannot assume that such corrections will not matter in the calculation of the average Lagrangian, and in fact they do. To this end, we have derived the average

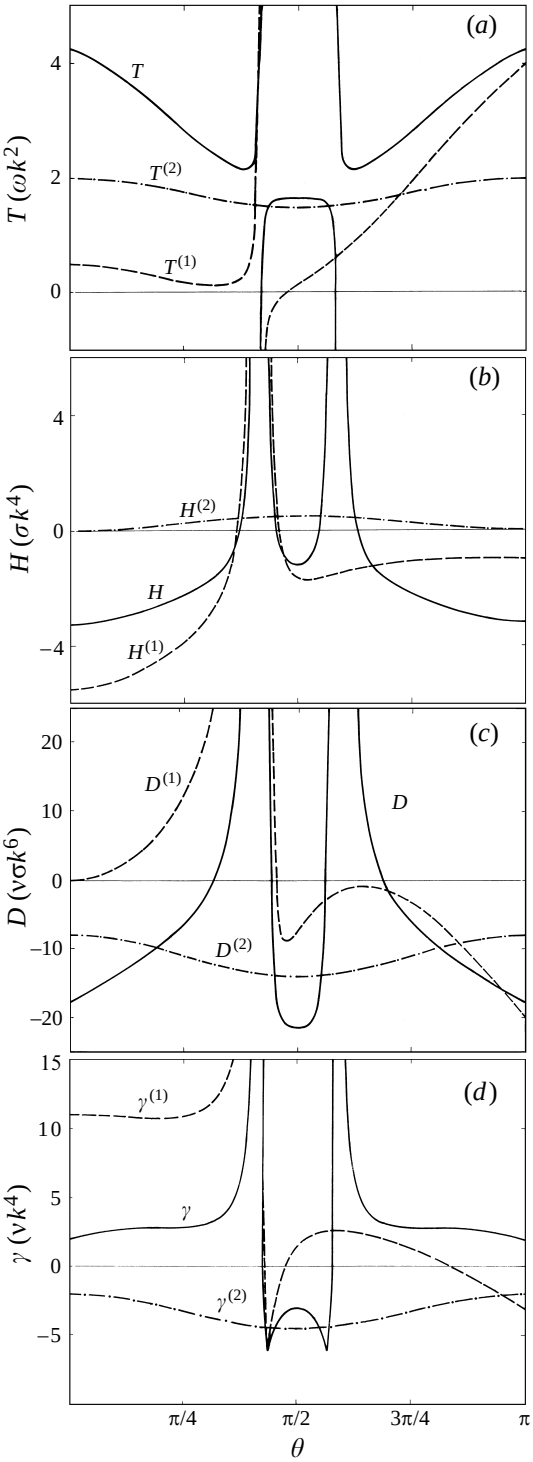

FIGURE 2. (*a*) The nonlinear detuning coefficients $T^{(1)}(\theta)$, $T^{(2)}(\theta)$, and $T(\theta)$ (unit $= \omega k^2$). (*b*) The mechanical energy coefficients $H^{(1)}(\theta)$, $H^{(2)}(\theta)$, and $H(\theta)$ (unit $= \sigma k^4$). (*c*) The energy dissipation function coefficients $D^{(1)}(\theta)$, $D^{(2)}(\theta)$, and $D(\theta)$ (unit $= \nu\sigma k^6$). (*d*) The nonlinear damping coefficients $\gamma^{(1)}(\theta)$, $\gamma^{(2)}(\theta)$, and $\gamma(\theta)$ (unit $= \nu k^4$).

Lagrangian (Appendix D, §D.5), which we write as follows:

$$\langle L\rangle = (\sigma k^2/\omega)[-\mathrm{i}\sum_j (a_j\dot{a}_j^* - a_j^*\dot{a}_j) + \omega\epsilon\sum_j (a_j a_{-j} + a_j^* a_{-j}^*) + \sum_{j,l} T_{jl}^{(1)}|a_l|^2|a_j|^2 + \sum_{j,l} T_{jl}^{(2)} a_l a_{-l} a_j^* a_{-j}^*]. \quad (86)$$

After a reduction to standing waves, we obtain

$$\langle L\rangle = 2N\sigma k^2[-(\mathrm{i}/\omega)(a\dot{a}^* - a^*\dot{a}) + \epsilon(a^2 + a^{*2}) + 2Nk^2\bar{T}(N)(|a|^2)^2]. \quad (87)$$

Using the relation (39) between the amplitude a and Miles' amplitudes p and q, and replacing $\bar{T}(N)$ by Miles' $C(N)$ (cf. (47)), Miles' expression (41)–(42), for the average Lagrangian is recovered. However, our Lagrangian has been derived from an expansion in terms of the variables $\dot{\zeta}$ and ϕ_ζ, and receives contributions from the terms $t_1 = \frac{1}{2}\rho\int \mathrm{d}^2x\ \dot{\zeta}_0\phi_2$ and $t_2 = \frac{1}{2}\rho\int \mathrm{d}^2x\ \dot{\zeta}_2\phi_0$ stemming from the kinetic energy, and the term $v_1 = \sigma\int \mathrm{d}^2x\nabla\zeta_2\cdot\nabla\zeta_0$ stemming from the potential energy. These terms contain $O(\epsilon^{3/2})$ resonant contributions (ζ_2 or ϕ_2) to the solutions. The terms t_1 and t_2 from the kinetic energy turn out to cancel (cf. Appendix D), but the term v_1 from the potential energy contributes to the Lagrangian. Thus, it is essential to keep the $O(\epsilon^{3/2})$ terms in order to derive the Lagrangian. It is therefore surprising that Miles obtains the correct Lagrangian, since he only considers contributions of $O(\epsilon)$.

4.2. *Nonlinear damping*

Here, we consider the damping coefficients to be added to the amplitude equation, (15). The relevant terms are given by (52). The coefficients $\gamma^{(0)}$, $\gamma_{jl}^{(1)}$, and $\gamma_{jl}^{(2)}$ are via (65)–(66) determined by the coefficients $H^{(0)}$, $H_{jl}^{(i)}$ for $\langle E\rangle$ (cf. (61)), and the coefficients $D^{(0)}$, $D_{jl}^{(i)}$ for $\langle D\rangle$ (cf. (62)). The calculation of these coefficients is carried out in detail in Appendices D and E.

For the mechanical energy, we recover (63) for $H^{(0)}$, while (64) for $H_{jl}^{(i)}$ is replaced by

$$H_{jl}^{(i)} = -(\sigma k^2/\omega)T_{jl}^{(i)} + h_{jl}^{(i)}, \quad (88)$$

where

$$h_{jl}^{(1)} = h^{(1)}(\theta) = [(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)(\tfrac{1}{2}d(b+6) - 4c) - 1]\sigma k^4, \quad (89)$$

$$h_{jl}^{(2)} = h^{(2)}(\theta) = 2(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)\sigma k^4; \quad (90)$$

d and b are given by (79)–(80). The counting restriction imposed on ζ_1 gives rise to a breakdown of the usual counting rule for the contribution $h^{(1)}(\theta)$, and thus for $H_{jk}^{(1)} = H^{(1)}(\theta)$,

$$h^{(1)}(0) \neq \tfrac{1}{2}\lim_{\theta\to 0} h^{(1)}(\theta), \quad H^{(1)}(0) \neq \tfrac{1}{2}\lim_{\theta\to 0} H^{(1)}(\theta). \quad (91)$$

As mentioned earlier, the breakdown of the usual counting rule occurs because we have neglected gravity. Taking this into account (Zhang & Viñals 1996*a*, *b*), it turns out that all terms not containing the factor $(1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)$, e.g. the term $-\sigma k^4$ in $h_{jl}^{(1)}$, should be multiplied by the factor $(1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)(1 + \tilde{\delta}_+ + \tilde{\delta}_-)$, where

$$\tilde{\delta}_+ \equiv \frac{1}{(kl_c)^2(1-c) + 1} \quad (92)$$

(recall that l_c is the capillary length, $l_c^2 = 2\sigma/\rho g$); $\tilde{\delta}_-$ is defined by replacing $1 - c$

with $1 + c$. In the capillary wave limit, we have $kl_c \gg 1$; hence, $\tilde{\delta}_+ \to \delta_+$, $\tilde{\delta}_- \to \delta_-$, and $(1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)(1 + \tilde{\delta}_+ + \tilde{\delta}_-) \to (1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)(1 + \delta_+ + \delta_-)1 = 1$.

In figure 2(*b*) $H_{jk}^{(1)}$ and $H_{jk}^{(2)}$ are shown, as well as $H(\theta)$, defined as

$$H(\theta) \equiv \tfrac{1}{2}[H^{(1)}(\theta) + H^{(2)}(\theta) + H^{(1)}(\pi - \theta) + H^{(2)}(\pi - \theta)]. \tag{93}$$

For standing waves,

$$\frac{1}{N}\sum_{l=1}^{N} H(\theta_{jl}) = \sigma k^4 \bar{H}(N). \tag{94}$$

For rolls ($N = 1$) we have from Milner $\bar{H}(1) = -\bar{T}_M(1) = -17/4$, where subindex M refers to the missing δ-functions in $\bar{T}(N)$. From Miles, $\bar{H}(1) = \hat{C}(1) = -1/8$. We find $\bar{H}(1) = -17/8$. For squares ($N = 2$) we obtain from Milner $\bar{H}(2) = -\bar{T}_M(2) = -(25 - \sqrt{2})/8 \simeq -2.95$. From Miles, $\bar{H}(2) = \hat{C}(2) = (3 - 2\sqrt{2})/16 \simeq 0.0107$. We find $\bar{H}(2) = -(1 + 18\sqrt{2})/16 \simeq -1.65$. For hexagons ($N = 3$) and the eightfold quasi-crystal ($N = 4$), we find $\bar{H}(3) = -(167 + 2592\sqrt{3})/2904 \simeq -1.60$, and $\bar{H}(4) \simeq -1.94$. We note that our energy is calculated directly from an expansion of (30), which is the same energy as Miles attempts to calculate. The disagreement between our results and the results of Miles is, as we see it, a consequence of Miles' omission of $O(\epsilon^{3/2})$ contributions to the solution for the surface displacement field. Also, it seems that Miles uses the opposite sign in the part of the potential energy that in our notation reads $v_2 = -\frac{1}{8}\sigma \int \mathrm{d}^2x(\nabla\zeta_0)^4$.

Next, consider the energy dissipation function D. At $O(\epsilon)$, we obtain (Appendix E)

$$D^{(0)} = 8\nu\sigma k^4. \tag{95}$$

From (63) and (65) we obtain the well-known result for bulk damping,

$$\gamma^{(0)} = 2\nu k^2, \tag{96}$$

which is also reported by Milner and by Miles (with $\omega\delta$ replacing $\gamma^{(0)}$). At $O(\epsilon^2)$, there is again not complete agreement. We find

$$D_{jl}^{(1)} = D^{(1)}(\theta) = 4\nu\sigma k^6[(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)(b(b + 9 - \tfrac{3}{2}c - c^2) - 3d + 17 - 10c - 9c^2) - 1], \tag{97}$$

$$D_{jl}^{(2)} = D^{(2)}(\theta) = -2\nu\sigma k^6(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)(7 - 3c^2). \tag{98}$$

Our results for $D^{(1)}(\theta)$ and $D^{(2)}(\theta)$ are shown in figure 2(*c*), Also shown is

$$D(\theta) \equiv \tfrac{1}{2}[D^{(1)}(\theta) + D^{(2)}(\theta) + D^{(1)}(\pi - \theta) + D^{(2)}(\pi - \theta)]. \tag{99}$$

Again,

$$D(\theta = 0) \neq \tfrac{1}{2}\lim_{\theta\to 0} D(\theta), \tag{100}$$

in the capillary wave limit. The $D_{jl}^{(i)}$ calculated by Milner are, to within the δ-functions that account for the values at $k = j$ and $k = -j$, in agreement with the $D_{jk}^{(i)}$ we find (Appendix E). For standing waves

$$\frac{1}{N}\sum_{l=1}^{N} D(\theta_{jl}) = \nu\sigma k^6 \bar{D}(N). \tag{101}$$

For rolls we find $\bar{D}(1) = -11$, and for squares we find $\bar{D}(2) = -(21+8\sqrt{2})/2 \simeq -16.16$.

In comparison, Miles (1993) finds $\bar{D}(1) = (4\omega/\nu k^2)\Gamma(1) = 20$, and $\bar{D}(2) = 47 + \sqrt{2} \simeq 48.41$. We have not been able to account fully for the nature of this disagreement, but the $O(\epsilon^{3/2})$ corrections to the neutral solution are still absent in Miles' (1994) calculation. The energy dissipation function is calculated via the velocity potential, in which terms that may stem from $O(\epsilon^{3/2})$ corrections seem to have been included, but Miles gives no particular values for $\Gamma(N)$ in that paper. For hexagons and the eightfold quasi-crystal, we obtain $\bar{D}(3) = -(34 + 64\sqrt{3})/121 \simeq -1.20$ and $\bar{D}(4) \simeq -11.29$, respectively.

Finally, we consider the values for the nonlinear damping coefficients $\gamma_{jl}^{(i)}$ given by (66). Our results for $\gamma_{jl}^{(i)} = \gamma^{(i)}(\theta)$, $i = 1, 2$, and for

$$\gamma(\theta) \equiv \tfrac{1}{2}[\gamma^{(1)}(\theta) + \gamma^{(2)}(\theta) + \gamma^{(1)}(\pi - \theta) + \gamma^{(2)}(\pi - \theta)], \tag{102}$$

are shown in figure 2(*d*). The same results for the damping coefficients were obtained using the alternative approach, (68). We note that the divergences at $\cos(\theta) = 2^{1/3} - 1$ are not removed to first order in viscosity (Zhang & Viñals 1997).

For standing waves we have

$$\frac{1}{N}\sum_{j=1}^{N} \gamma(\theta_{jk}) = \nu k^4 \bar{\gamma}(N), \tag{103}$$

where

$$\bar{\gamma}(N) = \tfrac{1}{4}\bar{D}(N) - 2\bar{H}(N). \tag{104}$$

Based on our calculations, we obtain

$$\bar{\gamma}(1) = 1.5, \quad \bar{\gamma}(2) \simeq -0.73, \quad \bar{\gamma}(3) \simeq 2.91, \quad \bar{\gamma}(4) \simeq 1.06. \tag{105}$$

For reasons already given above, the values reported by Milner ($\bar{\gamma}(1) = 3$, $\bar{\gamma}(2) \simeq 0.48$, and $\bar{\gamma}(3) \simeq 5.38$) and by Miles ($\bar{\gamma}(1) = 21/4 = 5.25$, $\bar{\gamma}(2) \simeq 12.08$) do not agree with the values we obtain.

5. Pattern formation

Below, we shall apply the amplitude equation for travelling waves, (15), supplemented with the damping terms, (52), as a basis for deriving an amplitude equation for standing waves (§5.1). From this, we shall particularly consider the selection of regular patterns (§5.2).

5.1. *Standing waves*

We apply the reduction method due to Riecke (1990). The amplitude equations for (a_j, a_{-j}^*) ((1) and its complex conjugate) are reduced to a single amplitude equation for standing waves. The idea is to use the neutral eigenmode at k_j with positive growth rate, and to eliminate the other mode, which is damped, in a Newell–Whitehead expansion of the amplitude equations for travelling waves.

The neutral solution to the amplitude equations for (a_j, a_{-j}^*) is derived on the assumption that all cubic terms in the amplitude equation may be neglected. The linearized equation for $\boldsymbol{u} = (a_j, a_{-j}^*)$ is then

$$(\partial_t + \boldsymbol{L}_0)\boldsymbol{u} = 0 \; ; \quad \boldsymbol{L}_0 = \begin{pmatrix} \gamma^{(0)} + \mathscr{G}_j & -\mathrm{i}c_0 f \\ \mathrm{i}c_0 f & \gamma^{(0)} + \mathscr{G}_{-j}^* \end{pmatrix}. \tag{106}$$

For a perturbation of the form

$$u = u_{\boldsymbol{q}} \exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{x}), \tag{107}$$

$\mathscr{G}_j$ can be replaced by $\mathrm{i}\delta\omega_j(\boldsymbol{q})$, and $\mathscr{G}^*_{-j} = -\mathscr{G}_j$. Moreover, the largest eigenvalue λ of $\boldsymbol{L}_0$ is

$$\lambda = -\gamma^{(0)} + [(c_0 f)^2 - (\delta\omega(\boldsymbol{q}))^2]^{1/2}. \tag{108}$$

The instability therefore occurs at $\delta\omega_j(\boldsymbol{q}) = 0$, and $f = f_c \equiv \gamma^{(0)}/c_0 = 8\nu\omega k$. It follows that $k\bar{f}_c = \gamma^{(0)}/\omega = 2\nu k^2/\omega$. An associated right eigenvector is $\boldsymbol{u}_r = (1, -\mathrm{i})$, hence the neutral solution for a Newell–Whitehead expansion is taken to be

$$\boldsymbol{u} = \boldsymbol{u}_r A_j. \tag{109}$$

The natural scaling is now (cf. (108)) $\boldsymbol{X} = \epsilon^{1/2}\boldsymbol{x}$, $T = \epsilon t$, and $\boldsymbol{u} = \epsilon^{1/2}\boldsymbol{u}_0 + \epsilon\boldsymbol{u}_1 + \ldots$. Recall that the expansion parameter is $\epsilon = k(\bar{f} - \bar{f}_c)$. The Newell–Whitehead expansion based on the above is carried out in detail in Appendix F. At $O(\epsilon^{3/2})$, the amplitude equation reads

$$0 = \dot{A}_j + (\gamma^{(0)} - c_0 f)A_j - \frac{1}{2\gamma^{(0)}}\mathscr{G}_j^2 A_j + \sum_l \gamma(\theta_{jl})|A_l|^2 A_j, \tag{110}$$

where $\mathscr{G}_j$ is given by (74). The same amplitude equation is also given by Milner (1991) (aside from a sign error in Milner's equation (26)). However, Milner also includes 'detuning' terms, which he uses in his analysis of the pattern formation. Including detuning (as in Riecke 1990) we were able to reproduce Milner's result aside from the term linear in $\hat{\boldsymbol{k}}_j \cdot \nabla$ which our calculations show should be absent.

The amplitude equation may be simplified on the assumptions of a regular pattern with $A_j = A_k = A$. If furthermore, $A = B\mathrm{e}^{\mathrm{i}\Phi}$, where B is real and Φ is stationary ($\dot{\phi} = 0$), the amplitude equation reads (omitting gradient terms)

$$0 = \dot{B} + (\gamma^{(0)} - c_0 f)B + 2N\nu k^4 \bar{\gamma}(N)B^3, \tag{111}$$

where $\bar{\gamma}(N)$ defined in (57) is introduced.

Equation (111) can be obtained directly from Miles' formulation, i.e. from the amplitude equation, (45), with the damping terms, (56), included. We go over to a description in terms of a real amplitude and phase, $a = B\mathrm{e}^{\mathrm{i}\Phi}$ (as in Ezerskii *et al.* 1986). On the assumption of a stationary phase, $\dot{\phi} = 0$ (anticipating a regular pattern), we obtain

$$\tfrac{1}{2}(\dot{a}\mathrm{e}^{-\mathrm{i}\Phi} + \dot{a}^*\mathrm{e}^{\mathrm{i}\Phi}) = \dot{B} = (-\gamma^{(0)} + c_0 f\sin(2\Phi))B - 2N\omega k^2\gamma(N)B^3, \tag{112}$$

$$\tfrac{1}{2}(\dot{a}\mathrm{e}^{-\mathrm{i}\Phi} - \dot{a}^*\mathrm{e}^{\mathrm{i}\Phi}) = 0 = c_0 f\cos(2\Phi)B + 2N\omega k^2 C(N)B^3. \tag{113}$$

From (113) we have $\cos(2\Phi) \sim B^2$. Hence, $1 - \sin(2\Phi) \sim B^4$, so to cubic order in B, $\sin(2\Phi) = 1$, and (111) follows from (112), recalling that $\omega\gamma(N) = \nu k^2\bar{\gamma}(N)$.

5.2. *Regular patterns*

We use the standing wave equation given in the previous subsection to study the selection of regular patterns in capillary wave systems. The amplitude B in (111) undergoes a pitchfork bifurcation when f crosses the value $f_c = \gamma^{(0)}/c_0$. For $N = 1$ (rolls), $N = 3$ (hexagons), and $N = 4$ (eightfold quasi-crystal), $\bar{\gamma}(N)$ is positive (cf. (105)), and the bifurcation is supercritical. For $N = 2$ (squares), the bifurcation is subcritical, and higher-order terms must be taken into account in order to examine the stability further (Christiansen *et al.* 1995). The theory predicts a 'first-order' transition from the zero solution $\zeta \equiv 0$ to the square pattern, in contrast to the 'second-order' transition implied by Milner's as well as Miles' positive value for $\bar{\gamma}(2)$. Furthermore, the 'first-order' transition takes place at a forcing $f = f_s$ below the critical value $f = f_c$. How far below

depends on the higher-order nonlinear coefficients, which have not been calculated. The fifth-order terms calculated by Milner only give partial contributions to the fifth-order coefficient of the standing wave amplitude equation – additional contributions originating from the travelling wave amplitude equation will occur. If we assume that the fifth-order coefficient in (111) has the form $2Nvk^6\bar{\tau}(N)$, with $\bar{\tau}(N)$ positive, the 'first-order' transition to a square pattern takes place at $f_s = (1-\Delta)f_c$, where (to fifth order)

$$\Delta \equiv \frac{3}{32}\frac{\bar{\gamma}(2)^2}{\bar{\tau}(2)}. \tag{114}$$

Above f_c, the square pattern is not necessarily the preferred pattern. To analyse this case, we follow standard procedures (see e.g. Cross & Hohenberg 1993), and consider the Lyapunov functional $\mathscr{F}_N[A_j, A_j^*]$ for standing waves. Assuming that the system is free to adjust its potential, we expect that the pattern corresponding to the deepest minimum of the functional $\mathscr{F}_N[A_j, A_j^*]$ will be selected.

First, consider the supercritical patterns. The amplitude equation correct to cubic order in A may be derived from a Lyapunov functional,

$$\mathscr{F}_N[A_j, A_j^*] = \int \mathrm{d}^2x \left[\sum_j (\gamma^{(0)} - c_0 f)|A_j|^2 + \frac{1}{2}\sum_{j,l}\gamma(\theta_{jl})|A_l|^2|A_j|^2 + \sum_j \frac{1}{2\gamma^{(0)}}|\mathscr{G}_j A_j|^2\right]. \tag{115}$$

On the assumption of a regular pattern with $A_j = A_l = A = B\mathrm{e}^{\mathrm{i}\Phi}$, we have the Lyapunov functional

$$\mathscr{F}_N[A_j = B\mathrm{e}^{\mathrm{i}\Phi}, A_j^* = B\mathrm{e}^{-\mathrm{i}\Phi}] = \int \mathrm{d}^2x\, F_N[B], \tag{116}$$

where

$$F_N[B] = (2N)[(\gamma^{(0)} - c_0 f)B^2 + Nvk^4\bar{\gamma}(N)B^4]. \tag{117}$$

The minimum is at $B = B_{min}$, where

$$B_{min}^2 = \frac{c_0 f - \gamma^{(0)}}{2Nvk^4\bar{\gamma}(N)}, \quad F_N[B_{min}] = -\frac{[c_0 f - \gamma^{(0)}]^2}{2vk^4\bar{\gamma}(N)}. \tag{118}$$

Hence, the selection principle translates into the question of minimizing the quantity $\bar{\gamma}(N)$ (see also Malomed, Nepomnyaschii & Tribelskii 1989, and Müller 1994). Based on this observation Milner concludes from his results that squares are selected. Miles' results, on the other hand, imply that rolls are more stable than squares. Our values of $\bar{\gamma}(N)$ for rolls ($N = 1$), hexagons ($N = 3$), and the eightfold quasi-crystal ($N = 4$) are given in (105). From these values we conclude that among these supercritical pattern, the quasi-crystalline pattern will be selected.

6. Comparison with experiments

As mentioned in the introduction, there are a number of high-aspect-ratio low-viscosity experiments on capillary waves, for which the theory reviewed here may be relevant (Ezerskii *et al.* 1985, 1986; Ezersky *et al.* 1994; Levin & Trubnikov 1986, Tufillaro *et al.* 1989 Christiansen *et al.* 1992, 1995 Bosch & van de Water 1993 Bosch 1995 Kudrolli & Gollub 1996 Binks & van de Water 1997). In this section

we shall be concerned with the applicability of the theory, and particularly point out the quantities that also seem important from an experimental perspective. The outcome of the experiments is the general formation of a square pattern, with new patterns in the form of a quasi-crystalline and a hexagonal pattern formed at very high frequencies (Christiansen *et al.* 1992; Bosch 1995). The consequences of our finding that the square pattern is subcritical is discussed in §6.1. We also consider various effects that play a role in the pattern selection in experimental systems. In §6.2 we examine finite-size effects, and in §6.3 we discuss effects arising when gravity or additional viscous terms become relevant.

6.1. *Subcriticality*

Consider the subcritical square pattern. The contributions to the fifth-order coefficient $\bar{\tau}(N)$, considered by Milner (1991), suggest that $\bar{\tau}(N)$ is of order $(\omega/\nu k^2)^2$, which is a large number for low-viscosity high-frequency experiments (Christiansen *et al.* 1995). Consequently, Δ defined by (114) should be very small. In the case $[(f/fc)-1] \gg \Delta$, the sixth-order Lyapunov integrand $F_2[B]$ for the square pattern has the minimum value

$$F_2[B_{min}] \simeq -(8/3)\nu[2\bar{\tau}(2)]^{-1/2}[(f/fc)-1]^{3/2}$$
$$= -\frac{32\sqrt{3}}{9}\nu[\bar{\gamma}(2)]^{-1}[\Delta]^{1/2}[(f/fc)-1]^{3/2}. \quad (119)$$

This value is numerically small compared to the values $F_N[B_{min}] = -2\nu[\bar{\gamma}(N)]^{-1} \times [(f/fc)-1]^2$ obtained in (118) for the supercritical patterns, and among those studied here the eightfold quasi-crystalline pattern will therefore be selected. This is consistent with observations by Christiansen *et al.* (1992) and by Bosch & van de Water (1993), where no transition value f_s below $f_c = 8\nu\omega k$ was observed. In contrast to this, Tufillaro *et al.* (1989) found a transition to a square pattern at a value $f_s = 0.67 f_c$ (Milner 1991), suggesting a smaller value of $\bar{\tau}(2)$ from their experiment. If $\bar{\tau}(N)$ is large in general (and not only for $N = 2$), as Milner's analysis suggests, the pattern selected will be the one with the smallest value of $\bar{\tau}(N)$.

6.2. *Finite-size effects*

A multiple-scales approach usually has one or more length scale ξ_i that diverges when $\epsilon \to 0$. This is also the case here. From the amplitude equation (110) it is clear that an important (coherence) length scale is

$$\xi_\perp \sim \frac{c_1}{\gamma^{(0)}}\epsilon^{-1/2} \sim \frac{\sigma}{\rho\nu\omega}\epsilon^{-1/2} \quad (120)$$

(there is also a diverging transversal length scale $\xi_\parallel \sim (c_3/\gamma^{(0)})^{1/2}\epsilon^{-1/4}$). In terms of the scaled variable $\mu\epsilon^{1/2}\boldsymbol{x}$ (see §3.1), the 'coherence length' $\xi_\perp$ is given by $\mu\epsilon^{1/2}\xi_\perp \sim \lambda$ (see also Edwards & Fauve 1994). Thus, when finite-size systems like laboratory experiments are considered, one generally expects to find a regime close to $\epsilon = 0$, where the pattern formation is controlled by the boundary conditions. Not until the diverging length scales ξ_i have values that are comparable to or smaller than the size of the system, may the amplitude equations be relevant. This is also the case for surface waves. For sufficiently small values of ϵ, a normal mode range determined by the boundary conditions is observed (see e.g. Christiansen *et al.* 1995). The range is, according to (120), smaller when the surface tension is smaller, or the density, viscosity, or frequency are larger. Experimentally, the maximal frequency is set by limits on the acceleration, $\nu\omega k \sim g$, giving a minimal coherence length $\xi_{\perp min} \sim [\sigma^4\rho^{-4}\nu^{-2}g^{-3}]^{1/5}\epsilon^{-1/2}$.

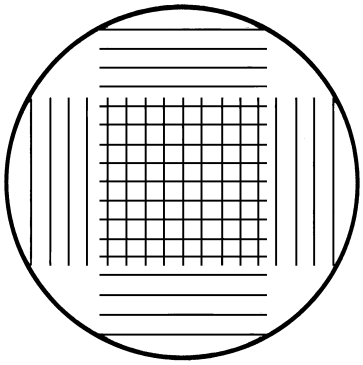

FIGURE 3. Schematic picture of square pattern formation in a circular cell when the coherence length $\xi_\perp$ is large (low viscosity).

At large frequencies, an eightfold quasi-crystalline pattern (and a hexagonal pattern) has been observed a few percent above the critical forcing, and prior to the formation of the square pattern (Christiansen *et al.* 1992; Bosch 1995). The observation requires that the normal mode range is sufficient narrow, i.e. that $\xi_\perp$ is comparable to the lateral dimension l of the fluid container, sufficiently close to the onset of surface waves. In other words the wavelength λ must be below a certain value λ_c to observe the quasi-crystalline pattern. The relation between λ_c and l has been considered in the experiment by Christiansen *et al.* (1992). At $l = 8.4$ cm, they report an aspect ratio $l/\lambda_c = 45$, which changes to the value $l/\lambda_c = 42$ at $l = 5.8$ cm. In comparison, we have from the expression for $\xi_\perp$ the relation $\lambda_c \propto l^{2/3}$, assuming that the value of ϵ at which the quasi-crystalline pattern is observed is frequency independent. This relation corresponds to a change in aspect ratio from 45 to 40.

Above the normal mode range, i.e. when $\xi_\perp$ becomes comparable to l, the surface pattern changes abruptly from a normal mode pattern determined by the boundaries to a stationary pattern determined by the nonlinear interactions. Both in low-viscosity and higher-viscosity experiments, the stationary pattern formed survives all the way to the rim, except for a distance of a wavelength or two, where the dynamics is spatiotemporally disordered. The length scale $\xi_\perp$ seems not to influence the size of the pattern! However, it has an influence on the form of the pattern, as a consequence of the spatial terms in the amplitude equation (110). Consider for example the square pattern in a circular cell. This consists of two perpendicular standing plane waves. If $\xi_\perp$ is small, the spatial terms in the amplitude equation can be neglected, and the square pattern extends over the entire region. If $\xi_\perp$ is large, it will not allow derivatives in the direction of the plane wave, and a pattern of the form schematically shown in figure 3 is enforced. The square pattern is seen only in a central region; outside this, only one (or none) of the plane waves has a non-negligible amplitude. Other regular patterns are formed in a similar way, with a central region outside which some of the amplitudes are negligible small. This explains in a simple way the form of the patterns observed in the low-viscosity experiments by Christiansen *et al.* (1992, 1995). It also explains why the patterns are observed to be quite independent of the form of the boundaries.

An interesting consequence of the spatial pattern variation discussed above is that the Lyapunov integrand no longer is a simple function $F_N[B]$. This may influence the pattern selection. In the simplest approximation (figure 3), the pattern forming region

may be divided into subregions i, each having a spatially independent Lyapunov integrand $F_{N^{(i)}}[B]$, where $N^{(i)}$ is the number of non-negligible amplitudes.

We emphasize that $\xi_\perp$ should not be interpreted as a length scale over which the amplitude smoothly changes in the direction of the plane wave, from the value B in the centre of the cell, to the value 0 at the rim. Such an amplitude behaviour is not seen experimentally. The amplitude essentially does not change until a couple of wavelengths away from the rim, at which point a spatiotemporal complex behaviour of the amplitude is encountered.

Above, we have only considered lateral finite-size effects. There are considerable changes in the linear and nonlinear theory when the fluid depth h is small, i.e. when kh becomes comparable to 1 or below. For capillary waves on shallow fluids, the linear dispersion relation changes, $\omega^2 = (\sigma/\rho)hk^4$ (see e.g. Zakharov *et al.* 1992), and the damping increases and becomes dominated by terms of order $(\nu\omega)^{1/2}/h$ rather than terms of order νk^2 (Kumar 1996). One may therefore expect changes in the pattern formation as well. For sufficiently thin fluid layers, the surface even oscillates in harmonic rather than subharmonic resonance with the driver (Kumar 1996; Müller *et al.* 1997).

6.3. *Gravitational and additional viscous effects*

We here focus on some effects the introduction of gravitation and additional viscous contributions have on the pattern selection. The dissipation arising from surface contamination and the dissipation from boundaries have been treated by Milner (1991), and their experimental relevance has been discussed by Christiansen *et al.* (1995). We do not examine these effects further here.

Consider the divergences of $\gamma(\theta)$ at $\theta = \theta^*$ $[0 \leqslant \theta^* \leqslant \frac{1}{2}\pi]$ and $\theta = \pi - \theta^*$, where $\cos(\theta^*) = 2^{1/3} - 1$. The positions of these divergences are of course crucial for the pattern selection. At the angle $\theta_{ij} = \theta^*$, we have $\omega(\boldsymbol{k}_i + \boldsymbol{k}_j) = \omega(\boldsymbol{k}_i) + \omega(\boldsymbol{k}_j)$. Thus $(\omega(\boldsymbol{k}) \to \omega(k))$,

$$\omega(2c_+k) = 2\omega(k), \tag{121}$$

where $c_+^2 = (1 + c)/2$. In the capillary wave limit, the solution $\theta = \theta_0$ to this equation is $\theta_0 = \theta^*$. However, when the gravitational term in the dispersion relation is taken into account,

$$\omega^2 = gk + \frac{\sigma}{\rho}k^3, \tag{122}$$

the solution is shifted to a smaller angle, $\theta_0 < \theta^*$ (Zhang & Viñals 1996, 1997). In terms of the dimensionless quantity kl_c ($l_c^2 = 2\sigma/\rho g$), (121) takes the form

$$c_+[1 + 2c_+^2(kl_c)^2] = 2 + (kl_c)^2. \tag{123}$$

The solution for c_+ decreases with increasing kl_c, from the value $c_+ = 1$ ($\theta_0 = 0$) at $kl_c = 1$, towards the value $c_+ = 2^{-1/3}$ ($\theta_0 = \theta^*$) for $kl_c \gg 1$. For $kl_c < 1$, $\gamma(\theta)$ has no divergence points.

Recent experiments (Kudrolli & Gollub 1996; Binks & van de Water 1997) have demonstrated that the pattern selection in a low-viscosity fluid is influenced by the shift of the divergence due to gravity. In the range where $kl_c \sim 1$, transitions have been observed from squares ($N = 2$) to hexagons ($N = 3$) (Kudrolli & Gollub 1996; Binks & van de Water 1997) and then to quasi-crystalline patterns ($N = 4$, $N = 5$) (Binks & van de Water 1997), where N is larger the closer kl_c is to unity, and the closer $\nu k^2/\omega$ is to zero.

While gravitation moves the divergence (triadic resonance) at $\theta = \theta^*$ towards smaller angles, the inclusion of higher-order viscous terms moves the divergence towards larger angles (see e.g. Edwards & Fauve 1994). The square pattern experimentally found at low viscosities and intermediate values of kl_c (Ezerskii *et al.* 1985, 1986; Ezersky *et al.* 1994; Levin & Trubnikov 1986, Tufillaro *et al.* 1989; Christiansen *et al.* 1992, 1995; Bosch & van de Water 1993; Bosch 1995; Kudrolli & Gollub 1996, Binks & van de Water 1997) is changed to rolls at higher viscosities (Fauve *et al.* 1992; Christiansen 1993; Edwards & Fauve 1994), presumably due to the shift of the peaks in $\gamma(\theta)$ towards $\pi/2$. In addition, the divergences will be changed to peaks of finite height when $O(\nu^2)$ terms are considered (Zhang & Viñals 1997).

Another gravitational effect that may be important is the $\tilde{\delta}$ contribution, discussed in §4.2, to $\gamma(\theta)$ when θ approaches zero or π. At $\theta = 0$, $\gamma(\theta)$ has a discontinuity. In the limit $kl_c \to \infty$, $\gamma(\theta \to 0) = 2$, and $\gamma(\theta = 0) = 1.5$. However, although large, kl_c is finite. As a consequence, $\gamma(\theta)$ will rapidly change very near $\theta = 0$, from the value 2 to the value 3, and then jump to the half value 1.5 at $\theta = 0$. The range $[0, \theta_\delta]$ where this rapid change takes place is given by (see (92))

$$(kl_c)^2[1 - \cos(\theta_\delta)] \sim 1. \tag{124}$$

Thus $\theta_\delta \sim (kl_c)^{-1}$. In comparison, we note that for the shift in peak position, $\theta^* - \theta_0 \sim (kl_c)^{-2}$. The non-zero value of θ_δ may therefore be more crucial for the pattern selection than the shift in peak position. Consider, for example, the high-aspect-ratio experiment by Ezerskii *et al.* (1985, 1986). From their data we find $kl_c = 4.5$. At this value, $\tilde{\delta}_+$ has only reduced its value to 0.15 at $\theta = \pi/4$! In the ethanol experiment by Christiansen *et al.* (1992) at $\omega/\pi = 380$ Hz, $kl_c = 9$, and $\tilde{\delta}_+ = 0.04$ at $\theta = \pi/4$. In the experiment by Bosch (1995), the value of kl_c is 7.5. Is seems that the parameter kl_c somehow has to be sufficiently large to observe the quasi-crystalline pattern.

7. Conclusions

It has been our aim to derive a consistent set of amplitude equations relevant to the pattern formation just above the onset of surface waves in driven capillary waves ($kl_c \gg 1$) at low viscosity ($\omega \gg \nu k^2$) and high aspect ratio ($kl \gg 1$, $kh \gg 1$, $\xi_\perp \gg l$). For this purpose, we have reviewed the Hamiltonian and the Lagrangian theories, and in detail we have constructed a Hamiltonian perturbation theory in the spirit of Milner's (1991) formulation. In this process, we have carefully pointed out the differences between our results for the nonlinear coefficients and the results obtained by Milner (1991) and Miles (1993), and by Zhang & Viñals (1996, 1997). Our calculations are carried out to first order in viscosity, and to $O((f - f_c)^{3/2})$, where f is the forcing amplitude, and f_c is its critical value.

From our standing waves analysis, we find that the square pattern is subcritical, and standard bifurcation theory predicts a 'first-order' transition to the square pattern, at a forcing $f_s < f_c$, in contrast to the 'second-order' transition implied by Milner's results. How far f_s lies below f_c depends on the higher-order nonlinear coefficients, which have not been calculated. However, theory (Milner 1991) suggests that $f_s \simeq f_c$, in agreement with some experiments (Christiansen *et al.* 1992; Bosch & van de Water 1993), and in disagreement with others (Tufillaro *et al.* 1989). Above f_c, the square pattern may no longer be selected. Again, this depends on the values of the higher-order nonlinear coefficients. However, if $f_s \simeq f_c$, the minimum of the Lyapunov functional for the square pattern is numerically small, and also for $f > f_c$, and the

most stable supercritical pattern may be selected. This may be an underlying reason for the observation of the eightfold quasi-crystalline pattern in the experiments by Christiansen *et al.* (1992) and by Bosch (1995).

Some effects must be considered when theory and experiments are compared. We have particularly considered finite-size effects and gravitational effects. In a finite-size system the coherence length (diverging at the Faraday instability) gives rise to a normal mode range. Based on available data, we have argued that the observed size-dependence of the selected wave pattern (Christiansen *et al.* 1992) may be related to the breakdown of this range (see also Edwards & Fauve 1994). Above the normal mode range, the presence of a finite coherence length $\xi_\perp$ does not reduce the size of the standing wave pattern, which seems to exist all the way to the rim of the fluid container except for a few wavelengths. It may, however, influence the form of the pattern, since a large coherence length will not allow amplitude derivatives in the plane wave directions.

The 'triadic resonance' in the $O(\nu)$ theory gives rise to a divergence of the nonlinear damping. This divergence is shifted when gravity or higher-order viscous terms are considered, and influences the pattern selection. In low-viscosity capillary gravity waves ($kl_c \sim 1$) transitions from squares to hexagons (Kudrolli & Gollub 1996; Binks & van de Water 1997) and then to quasi-crystalline patterns (Binks & van de Water 1997) have been observed. In high-viscosity fluids rolls are observed (Fauve *et al.* 1992; Christiansen 1993; Edwards & Fauve 1994). We have also identified another gravitational effect, which plays an important role in the nonlinear damping near zero wave interaction angle. Here, the value of the dimensionless capillary length kl_c is crucial. In the capillary wave experiments, the value is not always large, giving rise to noticeable changes to the nonlinear damping.

A correct theory of pattern selection near the Faraday instability rests on correctly derived amplitude equations. Motivated by the lack of a unique description, and by the discrepancies between results previously obtained, we have found it important to give a detailed description that allows others to follow our work. We hope that others will follow our example.

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Appendix A. Newell–Whitehead expansion for travelling waves

In this Appendix we shall give a detailed presentation of the Newell–Whitehead expansion of the hydrodynamical equations. The method of derivation is essentially that of Milner (1991).

A.1. *Expansion of* (7) *and* (8)

The goal is to solve the equation of motion, (7), and the kinematic surface condition, (8), on the free surface. This is done approximately by Taylor-expanding $\phi(\boldsymbol{x}, \zeta(\boldsymbol{x}))$ and its derivatives around $z = 0$,

$$\phi(\boldsymbol{x}, \zeta(\boldsymbol{x})) = \phi(\boldsymbol{x}, 0) + \zeta(\boldsymbol{x})\partial_z\phi(\boldsymbol{x}, 0) + \tfrac{1}{2}\zeta(\boldsymbol{x})^2\partial_z^2\phi(\boldsymbol{x}, 0) + \ldots, \tag{A 1}$$

with similar expansions for $\dot{\phi}(\boldsymbol{x}, \zeta(\boldsymbol{x}))$ and $\nabla\phi(\boldsymbol{x}, \zeta(\boldsymbol{x}))$. To third order in the fields ζ and ϕ, and with the vector notation $\boldsymbol{\Psi} \equiv (\zeta, \phi)$, we obtain from (7)–(8),

$$(\partial_t + \boldsymbol{L})\boldsymbol{\Psi}(\boldsymbol{x}, 0) = -\boldsymbol{N}, \tag{A 2}$$

where $\boldsymbol{L}$ in this Appendix denotes the linear operator

$$\boldsymbol{L} \equiv \begin{pmatrix} 0 & -\partial_z \\ -(\sigma/\rho)\nabla_\perp^2 - \omega_e^2 \bar{f}\cos(\omega_e t) & 0 \end{pmatrix}, \tag{A 3}$$

and $\boldsymbol{N}$ contains the nonlinear terms,

$$\boldsymbol{N} \equiv \begin{pmatrix} \zeta\nabla_\perp^2\phi + \frac{1}{2}\zeta^2\partial_z\nabla_\perp^2\phi + \nabla_\perp\zeta\cdot\nabla_\perp\phi + \zeta\nabla_\perp\zeta\cdot\partial_z\nabla_\perp\phi \\ (\sigma/2\rho)(\nabla_\perp^2\zeta)(\nabla_\perp\zeta)^2 + (\sigma/2\rho)\nabla_\perp\zeta\cdot\nabla_\perp(\nabla_\perp\zeta)^2 \\ \quad +\zeta\partial_z\partial_t\phi - \frac{1}{2}\zeta^2\nabla_\perp^2\partial_t\phi + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\zeta\partial_z(\nabla\phi)^2 \end{pmatrix}_{z=0} \tag{A 4}$$

(cf. Milner 1991). In writing the nonlinear terms, $\partial_z^2\phi$ has been replaced by $-\nabla_\perp^2\phi$. The two $\sigma/2\rho$ terms in $\boldsymbol{N}$ are obtained from an expansion of the pressure, (9), to $O(\zeta^3)$.

A.2. *Neutral modes*

We now perform a Newell–Whitehead expansion of (A 2). The first step in this expansion is to write down the neutral modes. With the abbreviations

$$f_j \equiv \exp(\mathrm{i}\boldsymbol{k}_j\cdot\boldsymbol{x}), \quad t_\omega \equiv \exp(-\mathrm{i}\omega t), \tag{A 5}$$

we have from §2.1,

$$\boldsymbol{\Psi}_0 = \begin{pmatrix} 1 \\ (-\mathrm{i}\omega/k)\mathrm{e}^{kz} \end{pmatrix} \sum_j a_j f_j t_\omega + \text{c.c.}, \tag{A 6}$$

where $\omega = \omega_e/2$ is related to $k = |\boldsymbol{k}_j|$ through the dispersion relation, (12). The amplitudes are constants in linear theory. In nonlinear theory, the amplitudes a_j are taken to be slowly varying functions in space ($\boldsymbol{X} = (X, Y)$) and time (T).

A.3. *Scaling of variables*

Next we scale our variables. The relevant expansion parameter is the control parameter $\epsilon = k(\bar{f} - \bar{f}_c)$. In the absence of damping $\bar{f}_c = 0$, and $\epsilon = k\bar{f}$.

As argued at the beginning of §4, we conclude from the shape of the resonance tongue (Landau & Lifshitz 1976; Cross & Hohenberg 1993) and the dispersion relation, (12), that the appropriate scaling of space and time variables are $\boldsymbol{X} = \epsilon\boldsymbol{x}$ and $T = \epsilon t$, while there is no slow variable in the z-direction, i.e.

$$\nabla_\perp \to \nabla_\perp + \epsilon\nabla_{\boldsymbol{x}}, \quad \partial_z \to \partial_z, \quad \partial_t \to \partial_t + \epsilon\partial_T. \tag{A 7}$$

The field $\boldsymbol{\Psi}$ is expanded in powers of $\epsilon^{1/2}$,

$$\boldsymbol{\Psi} = \epsilon^{1/2}\boldsymbol{\Psi}_0 + \epsilon\boldsymbol{\Psi}_1 + \epsilon^{3/2}\boldsymbol{\Psi}_2 + \ldots. \tag{A 8}$$

Thus $a_j(\boldsymbol{x}, t) \to \epsilon^{1/2}a_j(\boldsymbol{X}, T)$, which we will assume throughout the Appendices.

The linear operator $\boldsymbol{L}$ and the nonlinear vector $\boldsymbol{N}$ are expanded according to the above expansion,

$$\boldsymbol{L} = \boldsymbol{L}_0 + \epsilon^{1/2}\boldsymbol{L}_1 + \epsilon\boldsymbol{L}_2, \tag{A 9}$$

$$\boldsymbol{N} = \epsilon\boldsymbol{N}_1 + \epsilon^{3/2}\boldsymbol{N}_2, \tag{A 10}$$

where

$$\boldsymbol{L}_0 = \begin{pmatrix} 0 & -\partial_z \\ -(\omega^2/k^3)\nabla_\perp^2 & 0 \end{pmatrix}, \quad \boldsymbol{L}_1 = 0, \tag{A 11}$$

$$\boldsymbol{L}_2 = \begin{pmatrix} 0 & 0 \\ -2(\omega^2/k^3)\nabla_\perp \cdot \nabla_X - 4(\omega^2/k)\cos(2\omega t) & 0 \end{pmatrix}, \tag{A 12}$$

$$\boldsymbol{N}_1 = \begin{pmatrix} \zeta_0\nabla_\perp^2\phi_0 + \nabla_\perp\phi_0 \cdot \nabla_\perp\zeta_0 \\ \zeta_0\partial_z\partial_t\phi_0 + \frac{1}{2}(\nabla\phi_0)^2 \end{pmatrix}_{z=0}, \tag{A 13}$$

$$\boldsymbol{N}_2 = \begin{pmatrix} \zeta_0\nabla_\perp^2\phi_1 + \zeta_1\nabla_\perp^2\phi_0 + \frac{1}{2}\zeta_0^2\partial_z\nabla_\perp^2\phi_0 \\ \quad +\nabla_\perp\zeta_1 \cdot \nabla_\perp\phi_0 + \nabla_\perp\zeta_0 \cdot \nabla_\perp\phi_1 + \zeta_0\nabla_\perp\zeta_0 \cdot \partial_z\nabla\phi_0 \\ (\omega^2/2k^3)\nabla_\perp^2\zeta_0(\nabla_\perp\zeta_0)^2 + (\omega^2/2k^3)\nabla_\perp\zeta_0 \cdot \nabla_\perp(\nabla_\perp\zeta_0)^2 \\ \quad +\zeta_1\partial_z\partial_t\phi_0 + \zeta_0\partial_z\partial_t\phi_1 - \frac{1}{2}\zeta_0^2\nabla_\perp^2\partial_t\phi_0 \\ \quad +\nabla\phi_0 \cdot \nabla\phi_1 + \frac{1}{2}\zeta_0\partial_z(\nabla\phi_0)^2 \end{pmatrix}_{z=0} \tag{A 14}$$

Here, σ/ρ is replaced by ω^2/k^3.

When inserting the scaled variables in (A 1)–(A 2) and collecting terms of the same order, we obtain

$$(\partial_t + \boldsymbol{L}_0)\boldsymbol{\Psi}_0(\boldsymbol{x}, 0) = 0, \tag{A 15}$$

$$(\partial_t + \boldsymbol{L}_0)\boldsymbol{\Psi}_1(\boldsymbol{x}, 0) = -\boldsymbol{N}_1, \tag{A 16}$$

$$(\partial_t + \boldsymbol{L}_0)\boldsymbol{\Psi}_2(\boldsymbol{x}, 0) = -(\partial_T + L_2)\boldsymbol{\Psi}_0(\boldsymbol{x}, 0) - \boldsymbol{N}_2. \tag{A 17}$$

A.4. *From* $\boldsymbol{\Psi}_n(\boldsymbol{x}, 0)$ *to* $\boldsymbol{\Psi}_n(\boldsymbol{x}, z)$

From (A 15)–(A 17), the corrections $\boldsymbol{\Psi}_n(\boldsymbol{x}, 0)$ can be found, if one knows the corrections to order $< n$ for all z. We thus need to identify $\boldsymbol{\Psi}_n(\boldsymbol{x}, z)$, knowing $\boldsymbol{\Psi}_n(\boldsymbol{x}, 0)$. More precisely, we must determine $\phi_n(\boldsymbol{x}, z)$, only knowing $\phi_n(\boldsymbol{x}, 0)$. For this purpose, the incompressibility condition, (6), is used. This condition must also be expanded in orders of $\epsilon^{1/2}$ (Milner 1991),

$$\nabla^2\phi_0 = 0, \tag{A 18}$$

$$\nabla^2\phi_1 = 0, \tag{A 19}$$

$$\nabla^2\phi_2 = -2\nabla_\perp \cdot \nabla_X\phi_0. \tag{A 20}$$

The harmonic equations (A 18) and (A 19) are solved by multiplying terms of the form $\phi_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ in the solutions for $\phi_0(\boldsymbol{x}, 0)$ and $\phi_1(\boldsymbol{x}, 0)$ by a factor of $\mathrm{e}^{|\boldsymbol{k}|z}$. For example, (A 6) for $\boldsymbol{\Psi}_0(\boldsymbol{x}, z)$ is obtained in this way from the $z = 0$ solution $\boldsymbol{\Psi}_0(\boldsymbol{x}, 0) = \boldsymbol{\Psi}_r \sum_j a_j f_j t_\omega +$ c.c. (recall that $\boldsymbol{\Psi}_r = (1, -\mathrm{i}\omega/k)$ is a right eigenvector of the matrix $\bar{\boldsymbol{L}}_0$ defined in (11)).

For ϕ_2, however, we must first find a particular solution $\boldsymbol{\Psi}_2^{(p)} = (0, \phi_2^{(p)})$ to (A 20), and introduce it in (A 17). From (A 6) we have

$$-2\nabla_\perp \cdot \nabla_X\phi_0 = -2\omega\sum_j(\hat{\boldsymbol{k}}_j \cdot \nabla_X)a_j f_j t_\omega + \text{c.c.} \tag{A 21}$$

Hence, a particular solution to (A 20) is

$$\phi_2^{(p)} = -\frac{\omega z}{k}\mathrm{e}^{kz}\sum_j(\hat{\boldsymbol{k}}_j \cdot \nabla_X)a_j f_j t_\omega + \text{c.c.} \tag{A 22}$$

Now, we have

$$(\partial_t + \boldsymbol{L}_0)\boldsymbol{\Psi}_0 = 0 \equiv \boldsymbol{B}_0, \tag{A 23}$$

$$(\partial_t + \boldsymbol{L}_0)\boldsymbol{\Psi}_1 = -\boldsymbol{N}_1 \equiv \boldsymbol{B}_1, \tag{A 24}$$

$$(\partial_t + \boldsymbol{L}_0)\boldsymbol{\Psi}_2^{(h)} = -(\partial_t + \boldsymbol{L}_0)\boldsymbol{\Psi}_2^{(p)} - (\partial_T + \boldsymbol{L}_2)\boldsymbol{\Psi}_0 - \boldsymbol{N}_2 \equiv \boldsymbol{B}_2, \tag{A 25}$$

where $\boldsymbol{\Psi}_2^{(h)}$ is the harmonic correction (satisfying (6)), extended from the $z = 0$ solution in the way described above for $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_1$.

A.5. *Solvability conditions*

Solvability conditions for the above system of equations are expected at higher orders ($n \geqslant 1$). To derive the solvability conditions, we write $\boldsymbol{B}_n$ in a Fourier expansion form,

$$\boldsymbol{B}_n = \boldsymbol{B}_n^{(0)} + \sum_{m=1}^{\infty} \boldsymbol{B}_n^{(m)} t_{m\omega} + \text{c.c.} \tag{A 26}$$

A left eigenvector to the matrix $\bar{\boldsymbol{L}}_0$, (11), of the linear operator $\boldsymbol{L}_0$, associated with the eigenvalue $\mathrm{i}\omega$, is $\boldsymbol{\Psi}_l = (1, \mathrm{i}k/\omega)$. By multiplying this vector by (A 23)–(A 25), we have the solvability conditions

$$0 = \boldsymbol{\Psi}_l \cdot \boldsymbol{B}_n^{(1)}, \tag{A 27}$$

where it is understood, that we have extracted the part of $\boldsymbol{B}_n^{(1)}$ that represents modes associated with the frequency ω according to the dispersion relation, (12).

The problem is now completely defined within the framework of the Newell–Whitehead formalism, which can be solved order by order. With the present scaling, the solvability condition is trivially fulfilled for $n < 2$. For $n = 2$, we will have a solvability condition which gives the amplitude equation. In view of the solvability condition we shall be looking for nonlinear terms that combine in such a way that the sum of frequencies is ω, and the length k of the sum of wave vectors is given by the dispersion relation.

A.6. *Solution of (A 24)*

Consider (A 24) for $\boldsymbol{\Psi}_1$. To solve this, the four terms in $\boldsymbol{N}_1$ are evaluated. The contribution from these terms are of the form $\ldots a_j a_k^* f_j f_k^*$, or of the form $\ldots a_j a_k f_j f_k t_{2\omega}$ or its complex conjugate. The contributions are given in table 1 (where $c = c_{jl} = \hat{\boldsymbol{k}}_j \cdot \hat{\boldsymbol{k}}_l$). It follows that

$$\boldsymbol{B}_1 = \boldsymbol{B}_1^{(0)} + [\boldsymbol{B}_1^{(2)} t_{2\omega} + \text{c.c.}], \tag{A 28}$$

where

$$\boldsymbol{B}_1^{(0)} = \sum_{j,l} \begin{pmatrix} 0 \\ \omega^2(1-c) \end{pmatrix} a_j a_l^* f_j f_l^*, \tag{A 29}$$

$$\boldsymbol{B}_1^{(2)} = \sum_{j,l} \begin{pmatrix} -\mathrm{i}\omega k(1+c) \\ \frac{1}{2}\omega^2(3-c) \end{pmatrix} a_j a_l f_j f_l. \tag{A 30}$$

For simplicity, we shall use the notation

$$c_- = \frac{|\boldsymbol{k}_j - \boldsymbol{k}_l|}{2k} = \left(\frac{1-c}{2}\right)^{1/2}, \tag{A 31}$$

$$c_+ = \frac{|\boldsymbol{k}_j + \boldsymbol{k}_l|}{2k} = \left(\frac{1+c}{2}\right)^{1/2}. \tag{A 32}$$

Term	$\times a_j a_l^* f_j f_l^*$	$\times a_j a_l f_j f_l$
$N_{1,\zeta}$ $[\mathrm{i}\omega k]$		
$\zeta_0 \nabla_\perp^2 \phi_0$	0	1
$\nabla_\perp \zeta_0 \cdot \nabla_\perp \phi_0$	0	c
$\sum$	0	$1+c$
$N_{1,\phi}$ $[\omega^2]$		
$\zeta_0 \partial_z \partial_t \phi_0$	-2	-1
$\frac{1}{2}(\nabla\phi_0)^2$	$1+c$	$-\frac{1}{2}(1-c)$
$\sum$	$-(1-c)$	$-\frac{1}{2}(3-c)$

TABLE 1. Contributing terms from the vector $\boldsymbol{N}_1 = (N_{1,\zeta}, N_{1,\phi})$

From (A 24) we now have

$$\begin{aligned}\Psi_1(\boldsymbol{x},0) = &\sum_{j\neq l} \begin{pmatrix} 0 & -2kc_- \\ (2\omega^2/k)(1-c) & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \omega^2(1-c) \end{pmatrix} a_j a_l^* f_j f_l^* \\ &+ \sum_j \Psi_r b_j f_j t_\omega + \text{c.c.} \\ &+ \sum_{j,l} \begin{pmatrix} -2\mathrm{i}\omega - 2kc_+ \\ (2\omega^2/k)(1+c) - 2\mathrm{i}\omega \end{pmatrix}^{-1} \begin{pmatrix} -\mathrm{i}\omega k(1+c) \\ \frac{1}{2}\omega^2(3-c) \end{pmatrix} a_j a_l f_j f_l t_{2\omega} + \text{c.c.} + \ldots . \end{aligned} \tag{A 33}$$

By completing the matrix calculus, and extending the solution to all z as described above, one finds

$$\begin{aligned}\Psi_1(\boldsymbol{x},z) = &\frac{k}{2}\sum_{j,l}(1-\delta_{jl}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} a_j a_l^* f_j f_l^* + \begin{pmatrix} 1 \\ -(\mathrm{i}\omega/k)\mathrm{e}^{kz} \end{pmatrix} \sum_j b_j f_j t_\omega + \text{c.c.} \\ &+ \frac{k}{4}\sum_{j,l} \begin{pmatrix} d \\ -(i\omega/k)b\mathrm{e}^{|\boldsymbol{k}_j+\boldsymbol{k}_l|z} \end{pmatrix} a_j a_l f_j f_l t_{2\omega} + \text{c.c.} + \ldots , \end{aligned} \tag{A 34}$$

where b_j is an $O(\epsilon)$ neutral contribution to the amplitude a_j. The variables d and b are given in (79)–(80). Equation (A 34) for $\boldsymbol{\Psi}_1$ is in accordance with Milner's result, aside from a typing error in Milner's (A5) $[a_j^* a_l \to a_j a_l^*]$. The δ-function in (A 34) ensures that ζ_1 contains no 'zero-point' contribution $\sum_j |a_j|^2$, and arises as the capillary wave limit of $\tilde{\delta}$ defined by (92).

A.7. *The resonant terms of* $\boldsymbol{B}_2^{(1)}$

Next, we determine the resonant terms of $\boldsymbol{B}_2^{(1)}$. We first give the linear terms,

$$\begin{aligned}(\partial_t + L_0)\boldsymbol{\Psi}_2^{(p)}(\boldsymbol{x},0) &= \begin{pmatrix} -\partial_z \phi_2^{(p)}(\boldsymbol{x},0) \\ -\partial_t \phi_2^{(p)}(\boldsymbol{x},0) \end{pmatrix} \\ &= \begin{pmatrix} \omega/k \\ 0 \end{pmatrix} \sum_j (\hat{\boldsymbol{k}}_j \cdot \nabla_X a_j f_j t_\omega + \text{c.c.}, \end{aligned} \tag{A 35}$$

Term	$\times\lvert a_l\rvert^2 a_j f_j$		$\times a_l a_{-l} a^*_{-j} f_j$
$N_{2,\zeta}$ $[\frac{1}{2}\mathrm{i}\omega k^2]$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$	$\times 1$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$
$\zeta_0\nabla_\perp^2\phi_1$	$2(1+c)b$	0	0
$\zeta_1\nabla_\perp^2\phi_0$	$2-d$	-1	1
$\frac{1}{2}\zeta_0^2\partial_z\nabla_\perp^2\phi_0$	2	0	1
$\nabla_\perp\zeta_1\cdot\nabla_\perp\phi_0$	$(1+c)d-1+c$	0	-1
$\nabla_\perp\zeta_0\cdot\nabla_\perp\phi_1$	$-(1+c)b$	0	0
$\zeta_0\nabla_\perp\zeta_0\cdot\partial_z\nabla\phi_0$	$4c$	0	-2
$\sum$	$cd+(1+c)b+3+5c$	-1	-1
$N_{2,\phi}$ $[\frac{1}{2}\omega^2 k]$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$	$\times 1$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$
$(\omega^2/2k^3)\nabla_\perp^2\zeta_0(\nabla_\perp\zeta_0)^2$	-2	0	-1
$(\omega^2/2k^3)\nabla_\perp\zeta_0\cdot\nabla_\perp(\nabla_\perp\zeta_0)^2$	$-4c^2$	0	$-2c^2$
$\zeta_1\partial_z\partial_t\phi_0$	$-d-2$	1	-1
$\zeta_0\partial_z\partial_t\phi_1$	$-4c_+b$	0	0
$-\frac{1}{2}\zeta_0^2\nabla_\perp^2\partial_t\phi_0$	-6	0	-3
$\nabla\phi_0\cdot\nabla\phi_1$	$(1+c+2c_+)b$	0	0
$\frac{1}{2}\zeta_0\partial_z(\nabla\phi_0)^2$	$8(1+c)$	0	0
$\sum$	$-d+(1+c-2c_+)b-2+8c-4c^2$	1	$-5-2c^2$

TABLE 2. Contributing terms from the vector $\boldsymbol{N}_2=(N_{2,\zeta},N_{2,\phi})$

$$(\partial_T+L_2)\Psi_0(\boldsymbol{x},0)=\sum_j\begin{pmatrix}\partial_T a_j\\ -\dfrac{2\mathrm{i}\omega^2}{k^2}(\hat{\boldsymbol{k}}_j\cdot\nabla_X)a_j-\dfrac{2\omega^2}{k}a^*_{-j}-\dfrac{\mathrm{i}\omega}{k}\partial_T a_j\end{pmatrix}f_j t_\omega+\text{c.c.}\quad(\text{A }36)$$

The contributions to $\boldsymbol{B}_2^{(1)}$ from the 13 terms in $\boldsymbol{N}_2$ are given in table 2. To calculate these terms, we have used the 'reduction formula'

$$\sum_{j,l,m}g(j,l,m)a_j a_l a^*_m f_j f_l f^*_m\ \rightarrow$$
$$\sum_{j,l}(1-\tfrac{1}{2}\delta_+-\tfrac{1}{2}\delta_-)([g(j,l,l)+g(l,j,l)]|a_l|^2a_j+g(l,-l,-j)a_l a_{-l}a^*_{-j})f_j,\qquad(\text{A }37)$$

where $\delta_+=\delta_{jl}$ and $\delta_-=\delta_{-jl}$. The prefactor $(1-\frac{1}{2}\delta_+-\frac{1}{2}\delta_-)$ normally ensures the 'combinational rule' described in §4. However, the δ-function in ζ_1 breaks this rule. For $g(j,l,m)=\delta_{lm}$, 'reduction' yields

$$\sum_{j,l}(1-\tfrac{1}{2}\delta_+-\tfrac{1}{2}\delta_-)([1+\delta_+]|a_l|^2a_j+\delta_+a_l a_{-l}a^*_{-j})f_j$$
$$=\sum_{j,l}(1-\tfrac{1}{2}\delta_+-\tfrac{1}{2}\delta_-)(1+\delta_++\delta_-)|a_l|^2a_jf_j=\textstyle\sum_{j,l}|a_l|^2a_jf_j,\quad(\text{A }38)$$

which also follows directly by inserting $g(j,l,m)=\delta_{lm}$ on the left-hand side of (A 38). Below, the contributions arising from the δ-function in ζ_1 can be followed in detail as the terms without the prefactor $(1-\frac{1}{2}\delta_+-\frac{1}{2}\delta_-)$.

For $\boldsymbol{B}_2^{(1)}$ we get the expression

$$\boldsymbol{B}_2^{(1)} = -\sum_j \begin{pmatrix} \partial_T a_j + (\omega/k)(\hat{\boldsymbol{k}}_j \cdot \nabla_X) a_j \\ (-\mathrm{i}\omega/k)[\partial_T a_j - 2\mathrm{i}\omega a_{-j}^* + (2\omega/k)(\hat{\boldsymbol{k}}_j \cdot \nabla_X) a_j] \end{pmatrix} f_j$$
$$+\tfrac{1}{2}\mathrm{i}\omega k^2 \sum_{j,l} \begin{pmatrix} (1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)(\tau_\zeta^{(1)} |a_l|^2 a_j + \tau_\zeta^{(2)} a_l a_{-l} a_{-j}^*) + \tilde{\tau}_\zeta |a_l|^2 a_j \\ (-\mathrm{i}\omega/k)[(1 - \frac{1}{2}\delta_+ - \frac{1}{2}\delta_-)(\tau_\phi^{(1)} |a_l|^2 a_j + \tau_\phi^{(2)} a_l a_{-l} a_{-j}^*) + \tilde{\tau}_\phi |a_l|^2 a_j] \end{pmatrix} f_j, \tag{A 39}$$

where

$$\begin{aligned} \tau_\zeta^{(1)} &= -cd - (1+c)b - 3 - 5c \\ &= d - (2+c)b - 6 - 4c, && \text{(A 40)} \\ \tau_\zeta^{(2)} &= 1, \quad \tilde{\tau}_\zeta = 1, && \text{(A 41)} \\ \tau_\phi^{(1)} &= d - (1 + c - 2c_+)b + 2 - 8c + 4c^2 \\ &= 3d - (1+c)b - 2 - 12c + 4c^2, && \text{(A 42)} \\ \tau_\phi^{(2)} &= 5 + 2c^2, \quad \tilde{\tau}_\phi = -1. && \text{(A 43)} \end{aligned}$$

Here, we have applied the relations (79)–(80) between d and b.

A.8. *Amplitude equation*

From the solvability condition (A 27) for $n = 2$ we have

$$0 = \partial_T a_j - \mathrm{i}\omega a_{-j}^* + \frac{3\omega}{2k}(\hat{\boldsymbol{k}}_j \cdot \nabla_X) a_j - \mathrm{i}\sum_l T_{jl}^{(1)} |a_l|^2 a_j - \mathrm{i}\sum_l T_{jl}^{(2)} a_l a_{-l} a_{-j}^*, \tag{A 44}$$

where

$$\begin{aligned} T_{jl}^{(i)} &= \tfrac{1}{4}[(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)\tau_{jl}^{(i)} - \delta_{1i}\tilde{\tau}]\omega k^2, && \text{(A 45)} \\ \tau_{jl}^{(i)} &= \tau_\zeta^{(i)} + \tau_\phi^{(i)}, && \text{(A 46)} \\ \tilde{\tau} &= \tilde{\tau}_\zeta + \tilde{\tau}_\phi. && \text{(A 47)} \end{aligned}$$

We note that $\tilde{\tau}$ is zero, and hence the 'combinational rule' (§4) holds for $T_{jk}^{(i)}$ (cf. (83)). Inserting the values for $\tau_{jl}^{(i)}$ into (A 45), equations (75)–76) for $T_{jl}^{(i)}$ are produced.

Returning to unscaled units ($\epsilon\nabla_X \to \nabla_\perp$, $\epsilon\partial_T \to \partial_t$, $\epsilon^{1/2} a_j \to a_j$), the amplitude equation (15) is derived with $c_0 = k/4\omega$, and $\mathscr{G}_j$ given by (74) ($c_1 = 3\omega/2k$).

A.9. *Results for* $\boldsymbol{\Psi}_2$

Using the amplitude equation to replace $\partial_T a_j$ in the expression (A39) for $\boldsymbol{B}_2^{(1)}$, we obtain

$$\boldsymbol{B}_2^{(1)} = \boldsymbol{\Psi}_r^* \Big(\sum_j [-\mathrm{i}\omega a_{-j}^* + \tfrac{1}{2}(\hat{\boldsymbol{k}}_j \cdot \nabla_X) a_j] - \tfrac{1}{2} i\omega k^2 \textstyle\sum_{j,l} [\mathscr{B}_{jl}^{(1)} |a_l|^2 a_j + \mathscr{B}_{jl}^{(2)} a_l a_{-l} a_{-j}^*]\Big) f_j, \tag{A 48}$$

where

$$\begin{aligned} \mathscr{B}_{jl}^{(1)} &= \tfrac{1}{2}(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)(\tau_\phi^{(1)} - \tau_\zeta^{(1)}) + \tfrac{1}{2}(\tilde{\tau}_\phi - \tilde{\tau}_\zeta) \\ &= (1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)[d + \tfrac{1}{2}b + 2(1-c)^2] - 1, && \text{(A 49)} \end{aligned}$$

$$\mathscr{B}^{(2)}_{jl} = \tfrac{1}{2}(1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)(\tau^{(2)}_\phi - \tau^{(2)}_\zeta)$$
$$= (1 - \tfrac{1}{2}\delta_+ - \tfrac{1}{2}\delta_-)[2 + c^2]. \quad \text{(A 50)}$$

In order to find the resonant contribution to $\boldsymbol{\Psi}_2$, we note that $\boldsymbol{\Psi}^*_r = (1, \mathrm{i}\omega/k)$ is a right eigenvector to $\bar{\boldsymbol{L}}_0$ associated with the eigenvalue $-\mathrm{i}\omega$ (or eigenvalue $-2\mathrm{i}\omega$ for $(-\mathrm{i}\omega + \bar{\boldsymbol{L}}_0)$). Hence,

$$\boldsymbol{\Psi}_2(\boldsymbol{x}, 0) = \left[\sum_j \boldsymbol{\Psi}_r c_j f_j + \frac{\mathrm{i}}{2\omega}\boldsymbol{B}^{(1)}_2\right] t_\omega + \text{c.c.} + \ldots, \quad \text{(A 51)}$$

where c_j is an $O(\epsilon^{3/2})$ neutral contribution to the amplitude a_j. Inserting (A 48) for $\boldsymbol{B}^{(1)}_2$ into (A 51), and extending this solution to all z yields

$$\boldsymbol{\Psi}^{(h)}_2(\boldsymbol{x}, z) = \begin{pmatrix} 1 \\ -(\mathrm{i}\omega/k)\mathrm{e}^{kz} \end{pmatrix} \sum_j c_j f_j t_w + \text{c.c.}$$
$$+ \begin{pmatrix} 1 \\ (\mathrm{i}\omega/k)\mathrm{e}^{kz} \end{pmatrix} \left(\sum_j \left(\frac{1}{2} a^*_{-j} + \frac{\mathrm{i}}{4k}(\hat{\boldsymbol{k}}_j \cdot \nabla_X) a_j \right) f_j t_\omega \right.$$
$$\left. + \frac{k^2}{4} \sum_{j,l} [\mathscr{B}^{(1)}_{jl} |a_l|^2 a_j + \mathscr{B}^{(2)}_{jl} a_l a_{-l} a^*_{-j}] f_j t_\omega \right) + \text{c.c.} + \ldots. \quad \text{(A 52)}$$

Our result for $\boldsymbol{\Psi}_2$ is in accordance with Milner's result, aside from the (important!) δ-functions, which are miscounted in Milner's (B 5), and aside from a typing error in Milner's (B 4) ($4\omega k^2 \to k^2/4$).

Appendix B. Expansion of the Dirichlet–Neumann operator

In this Appendix we expand the Dirichlet–Neumann operator $\hat{G}[\zeta]$ in orders of ζ (see e.g. Craig & Sulem 1993).

B.1. *Expansion of the operator* $\hat{G}[\zeta]$

To expand the operator $\hat{G}[\zeta]$, we use its definition, (17),

$$\hat{G}[\zeta]\phi(\boldsymbol{x}, z) = [\hat{D} - \nabla_\perp \zeta \cdot \nabla_\perp]\phi(\boldsymbol{x}, z), \quad \text{(B 1)}$$

where ∂_z is replaced by the z-independent operator $\hat{D}$ that multiplies terms of the form $\phi_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ in the Fourier expansion of ϕ by $|\boldsymbol{k}|$. In this replacement, the harmonic equation for ϕ, (6), has been invoked, and formally $\hat{D}$ can be expressed as $\hat{D} = (-\nabla^2_\perp)^{1/2}$.

We now write

$$\hat{G}[\zeta] = \sum_m \hat{G}_m[\zeta], \quad \text{(B 2)}$$

where $\hat{G}_m$ is of mth order in ζ. Replacing z by ζ in (B 1), and expanding ϕ and its derivatives around $z = 0$ (cf. (A 1)), we obtain the following operator equality:

$$\sum_m \hat{G}_m \left(\sum_n \frac{1}{n!}\zeta^n \hat{D}^n \right) = \sum_m \sum_n \frac{1}{n!}\hat{G}_m(\zeta^n \hat{D}^n) = \sum_n \frac{1}{n!}\zeta^n [\hat{D} - \nabla_\perp \zeta \cdot \nabla_\perp]\hat{D}^n. \quad \text{(B 3)}$$

B.2. *Derivation of* $\hat{G}_m$

From (B3), $\hat{G}_m$ can be derived order by order. To second order in ζ, we have

$$\hat{G}_0 = \hat{D}, \tag{B 4}$$

$$\hat{G}_1 = -\hat{G}_0(\zeta\hat{D}) - \nabla_\perp\zeta\cdot\nabla_\perp + \zeta\hat{D}^2 = -\hat{D}\zeta\hat{D} - \nabla_\perp\zeta\cdot\nabla_\perp - \zeta\nabla_\perp^2, \tag{B 5}$$

$$\begin{aligned}\hat{G}_2 &= -\hat{G}_1(\zeta\hat{D}) - \tfrac{1}{2}\hat{G}_0(\zeta^2\hat{D}^2) - \zeta\nabla_\perp\zeta\cdot\nabla_\perp\hat{D} + \tfrac{1}{2}\zeta^2\hat{D}^3 \\ &= \hat{D}\zeta\hat{D}\zeta\hat{D} + \tfrac{1}{2}\hat{D}\zeta^2\nabla_\perp^2 + \tfrac{1}{2}\nabla_\perp^2\zeta^2\hat{D}.\end{aligned} \tag{B 6}$$

In the equation for $\hat{G}_2$, we have used the formula

$$\nabla_\perp^2\zeta^2\hat{D} = 2\nabla_\perp\zeta\cdot\nabla_\perp\zeta\hat{D} + 2\zeta\nabla_\perp^2\zeta\hat{D} - 2\zeta\nabla_\perp\zeta\cdot\nabla_\perp\hat{D} - \zeta^2\nabla_\perp^2\hat{D}. \tag{B 7}$$

The expanded form of the Dirichlet–Neumann operator can now be inserted into the equations of motion for the surface fields ζ and ϕ_ζ. The equations of motion expanded to third order are now easily obtained (Zhang & Viñals 1996, 1997). The resulting equations are those used by Zhang & Viñals in their multiple-scales analysis.

Appendix C. The energy dissipation formula

In this Appendix we consider the dissipation formula introduced in §3.2 in order to facilitate the calculation of the damping coefficients $\gamma^{(0)}$, $\gamma_{jl}^{(1)}$, and $\gamma_{jl}^{(2)}$. The damping formalism described here is an extension of the formalism described in Landau & Lifshitz (1987).

The total mechanical energy of the system is $E = \int \mathrm{d}^2x\ [\mathscr{T} + \mathscr{V}]$, where $\mathscr{T}$ and $\mathscr{V}$ are given by (28). The rate of change of this mechanical energy, $\dot{E}$, will contain a term, D, which reflects that the rate of change of the mechanical energy is mainly balanced by viscous stresses (Landau & Lifshitz 1987). We here show that in the presence of the time-dependent forcing term in the mechanical energy, there will, in addition, be a balancing term in $\dot{E}$ which reflects the presence of the forcing term, i.e. a term proportional to f.

To make the calculation tractable, we consider the rate of change of the kinetic energy, $\partial_t \int \mathrm{d}^2x\ \mathscr{T}$, and the rate of change of the potential energy, $\partial_t \int \mathrm{d}^2x\ \mathscr{V}$, separately.

C.1. *Kinetic energy*

For the kinetic energy part we have

$$\begin{aligned}\partial_t \int \mathrm{d}^2x\ \mathscr{T} &= \int \mathrm{d}^2x\ \partial_t \int_{-h}^{\zeta} \mathrm{d}z\ [\tfrac{1}{2}\rho\boldsymbol{v}^2] \\ &= \int \mathrm{d}^2x\ [\tfrac{1}{2}\rho\boldsymbol{v}_\zeta^2\dot{\zeta}] + \int \mathrm{d}^2x\ \textstyle\int_{-h}^{\zeta}\mathrm{d}z\ [\tfrac{1}{2}\rho\partial_t(\boldsymbol{v}^2)],\end{aligned} \tag{C 1}$$

where subindex ζ refers to the surface $z = \zeta$. To calculate the last term, we apply the Navier–Stokes equation for incompressible fluids (Landau & Lifshitz 1987),

$$\rho(\partial_t v_i + v_k\partial_k v_i) = -\partial_i p + \eta\partial_k(\partial_i v_k + \partial_k v_i) - \partial_i U, \tag{C 2}$$

where $U = -\rho f\cos(\omega_e t)z$ is the effective gravitational potential in the capillary wave limit. By simple vector multiplication with $\boldsymbol{v}$ we have

$$\begin{aligned}\frac{\rho}{2}\partial_t\boldsymbol{v}^2 &= -v_i\partial_i(p + \tfrac{1}{2}\rho\boldsymbol{v}^2 + U) + \eta v_i\partial_k(\partial_i v_k + \partial_k v_i) \\ &= -\partial_i[v_i(p + \tfrac{1}{2}\rho\boldsymbol{v}^2 + U)] + \eta\partial_k[v_i(\partial_i v_k + \partial_k v_i)] - \eta(\partial_k v_i)(\partial_i v_k + \partial_k v_i),\end{aligned} \tag{C 3}$$

where we have used the incompressibility condition, $\partial_i v_i = 0$. Inserting (C 3) into (C 1) and applying Green's formula we obtain

$$\partial_t \int \mathrm{d}^2x\, \mathscr{T} = \int \mathrm{d}^2x\, [\tfrac{1}{2}\rho \boldsymbol{v}_\zeta^2 \dot{\zeta}] + D \\ - \oint \mathrm{d}S\, (v_i)_\zeta [(p_\zeta + \tfrac{1}{2}\rho \boldsymbol{v}_\zeta^2 + U_\zeta) n_i - \eta((\partial_i v_k)_\zeta + (\partial_k v_i)_\zeta) n_k], \quad \text{(C 4)}$$

where $\boldsymbol{n}$ is a unit vector normal to the surface, $\mathrm{d}S$ is a surface element, and D is defined by (60).

On the free surface, we have

$$\boldsymbol{n} = \frac{(-\nabla_\perp \zeta, 1)}{g^{1/2}}, \quad \mathrm{d}S = g^{1/2} \mathrm{d}^2x, \quad \text{(C 5)}$$

where the determinant of the metrical tensor, g, is given in (29). Consequently,

$$\boldsymbol{v}_\zeta \cdot \boldsymbol{n} \mathrm{d}S = \boldsymbol{v}_\zeta \cdot (-\nabla_\perp \zeta, 1) \mathrm{d}^2x = \dot{\zeta} \mathrm{d}^2x, \quad \text{(C 6)}$$

(which is the kinematic surface condition). Thus the two $\frac{1}{2}\rho \boldsymbol{v}_\zeta^2$ terms cancel. Moreover,

$$p_\zeta n_i = \eta((\partial_i v_k)_\zeta + (\partial_k v_i)_\zeta) n_k + p_\zeta^{(ideal)} n_i, \quad \text{(C 7)}$$

where $p_\zeta^{(ideal)}$ is the ideal fluid result, (9) (see e.g. Landau & Lifshitz 1987).

The final result for $\partial_t \int \mathrm{d}^2x\, \mathscr{T}$ is

$$\partial_t \int \mathrm{d}^2x\, \mathscr{T} = D - \int \mathrm{d}^2x\, [p_\zeta^{(ideal)} + U_\zeta] \dot{\zeta}. \quad \text{(C 8)}$$

C.2. *Potential energy*

The rate of change of the potential energy may be written as

$$\partial_t \int \mathrm{d}^2x\, \mathscr{V} = \int \mathrm{d}^2x\, \partial_t \left(\int_0^\zeta \mathrm{d}z\, [-\rho f \cos(\omega_e t) z] + \sigma(g^{1/2} - 1) \right) \\ = \int \mathrm{d}^2x\, [-\rho f \cos(\omega_e t) \zeta \dot{\zeta}] \\ + \int \mathrm{d}^2x \int_0^\zeta \mathrm{d}z\, [\rho f \omega_e \sin(\omega_e t) z] + \int \mathrm{d}^2x\, \frac{\sigma}{g^{1/2}} \nabla_\perp \zeta \cdot \nabla_\perp \dot{\zeta}. \quad \text{(C 9)}$$

In the capillary wave limit we have $U_\zeta = -\rho f \cos(\omega_e t)\zeta$. For the last term we use partial integration to get $p_\zeta^{(ideal)} \dot{\zeta}$. We thus obtain,

$$\partial_t \int \mathrm{d}^2x\, \mathscr{V} = \int \mathrm{d}^2x\, \left([p_\zeta^{(ideal)} + U_\zeta] \dot{\zeta} + \tfrac{1}{2}\rho f \omega_e \sin(\omega_e t) \zeta^2 \right). \quad \text{(C 10)}$$

C.3. *Complete calculation*

The calculation may now be completed. By combining (C 8) and (C 10), the energy dissipation formula given by (59) is obtained.

Appendix D. Expansion of the mechanical energy and the Lagrangian

In this Appendix we consider the mechanical energy and the Lagrangian for undamped surface waves in greater detail.

D.1. *Kinetic energy*

First, consider the kinetic energy $\int \mathrm{d}^2x\ \mathscr{T}$, where $\mathscr{T}$ is given in (28). Invoking the incompressibility condition, $\boldsymbol{v}^2 = (\nabla\phi)^2$ can be rewritten,

$$(\nabla\phi)^2 = \nabla\cdot(\phi\nabla\phi). \tag{D 1}$$

Applying Green's formula, we have

$$\int \mathrm{d}^2x\ \mathscr{T} = \frac{\rho}{2}\int \mathrm{d}^3x\ \nabla\cdot(\phi\nabla\phi) = \frac{\rho}{2}\oint \phi_\zeta\nabla\phi_\zeta\cdot\boldsymbol{n}\mathrm{d}S, \tag{D 2}$$

where $\boldsymbol{n}$ is a unit vector normal to the surface, and $\mathrm{d}S$ is the surface element. Replacing $\boldsymbol{n}\mathrm{d}S$ by $(-\nabla_\perp\zeta, 1)\mathrm{d}^2x$ (cf. (C 5)), we have

$$\int \mathrm{d}^2x\ \mathscr{T} = \frac{\rho}{2}\int \mathrm{d}^2x\ \phi_\zeta\nabla\phi_\zeta\cdot(-\nabla_\perp\zeta, 1) = \frac{\rho}{2}\int \mathrm{d}^2x\ \phi_\zeta\dot{\zeta}, \tag{D 3}$$

where we have used the kinematic surface condition, (8). In passing from (D2) to (D3) we have used that the velocity field vanishes at the walls of the container.

D.2. *Mechanical energy expanded*

We shall insert the Taylor expansion for ϕ_ζ, (A1), in the expression for $\mathscr{T}$. Moreover $g^{1/2}$ is expanded as,

$$g^{1/2} - 1 = \tfrac{1}{2}(\nabla_\perp\zeta)^2 - \tfrac{1}{8}(\nabla_\perp\zeta)^4 + \ldots. \tag{D 4}$$

For the mechanical energy, we obtain

$$\begin{aligned} E = {} & \frac{\rho}{2}\int \mathrm{d}^2x\ \dot{\zeta}[\phi(\boldsymbol{x},0) + \zeta\partial_z\phi(\boldsymbol{x},0) + \tfrac{1}{2}\zeta^2\partial_z^2\phi(\boldsymbol{x},0)] \\ & -\frac{\rho}{2}\int \mathrm{d}^2x\ f\cos(2\omega t)\zeta_0^2 + \sigma\int \mathrm{d}^2x\ [\tfrac{1}{2}(\nabla_\perp\zeta)^2 - \tfrac{1}{8}(\nabla_\perp\zeta)^4] + \ldots. \end{aligned} \tag{D 5}$$

D.3. *Mechanical energy in scaled variables*

Introducing the slow scales $\boldsymbol{X} = \epsilon x$ and $T = \epsilon t$, and the expansion (A8) for $\boldsymbol{\Psi}$, we obtain the expansion

$$E = \epsilon E_1 + \epsilon^2 E_2, \tag{D 6}$$

where

$$E_1 = \frac{\rho}{2}\int \mathrm{d}^2x\ [(\partial_t\zeta_0)\phi_0] + \sigma\int \mathrm{d}^2x\ [\tfrac{1}{2}(\nabla\zeta_0)^2], \tag{D 7}$$

$$\begin{aligned} E_2 = {} & \frac{\rho}{2}\int \mathrm{d}^2x\ [(\partial_t\zeta_0)\phi_2 + (\partial_t\zeta_0)\zeta_0\partial_z\phi_1 + (\partial_t\zeta_0)\zeta_1\partial_z\phi_0 + \tfrac{1}{2}(\partial_t\zeta_0)\zeta_0^2\partial_z^2\phi_0 \\ & +(\partial_t\zeta_1)\phi_1 + (\partial_t\zeta_1)\zeta_0\partial_z\phi_0 + (\partial_t\zeta_2)\phi_0 + (\partial_T\zeta_0)\phi_0] \\ & -2\frac{\rho\omega^2}{k}\int \mathrm{d}^2x\ \cos(2\omega t)\zeta_0^2 \\ & +\sigma\int \mathrm{d}^2x\ [\tfrac{1}{2}(\nabla\zeta_1)^2 + \nabla\zeta_0\cdot\nabla\zeta_2 + \nabla\zeta_0\cdot\nabla_X\zeta_0 - \tfrac{1}{8}(\nabla\zeta_0)^4] + \ldots \end{aligned} \tag{D 8}$$

(every term evaluated at $z = 0$). Based on the results from Appendix A, the average energy $\langle E\rangle$, (61), can be derived. As noted in the beginning of at §3, we do not include

Term	$\times\lvert a_l\rvert^2\lvert a_j\rvert^2$	$\times a_l a_{-l} a_j^* a_{-j}^*$	
E_2 $[\sigma k^4]$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$	$\times 1$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$
$\frac{1}{2}\rho(\partial_t\zeta_0)\zeta_0\partial_z\phi_1$	$c_+ b$	0	0
$\frac{1}{2}\rho(\partial_t\zeta_0)\zeta_1\partial_z\phi_0$	$1-\frac{1}{2}d$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{4}\rho(\partial_t\zeta_0)\zeta_0^2\partial_z^2\phi_0$	1	0	$\frac{1}{2}$
$\frac{1}{2}\rho(\partial_t\zeta_1)\phi_1$	$\frac{1}{4}db$	0	0
$\frac{1}{2}\rho(\partial_t\zeta_1)\zeta_0\partial_z\phi_0$	d	0	0
$\sum$	$\frac{1}{4}d(b+2)+c_+b+2$	$-\frac{1}{2}$	1
$\frac{1}{2}\sigma(\nabla\zeta_1)^2$	$\frac{1}{4}(1+c)d^2+\frac{1}{4}(1-c)$	0	$\frac{1}{4}$
$\sigma\nabla\zeta_0\cdot\nabla\zeta_2$	$\frac{1}{2}d+\frac{1}{4}b+(1-c)^2$	$-\frac{1}{2}$	$\frac{1}{2}(2+c^2)$
$-\frac{1}{8}\sigma(\nabla\zeta_0)^4$	$-\frac{1}{2}(1+2c^2)$	0	$-\frac{1}{4}(1+2c^2)$
$\sum$	$\frac{1}{4}[(1+c)d^2+2d+b+3-9c]$	$-\frac{1}{2}$	1

TABLE 3. Contributing terms of $O(\epsilon^2)$ in the expansion of the kinetic and potential energies. All terms proportional to c vanish in the last column (use $l\to -l$)

the higher-order resonant corrections (b_j, c_j) to the amplitude field in our derivation of the mechanical energy E. From (D 7) the result for $H^{(0)}$, defined in (61), is obtained. Both terms in E_1 give the contribution σk^2 to $H^{(0)}$; thus $H^{(0)}$ is correctly given by (63).

Next, we consider the terms in E_2. To calculate these terms, we have used the following 'reduction formula':

$$\sum_{j,k,l,m} g(j,k,l,m)a_j a_k a_l^* a_m^* f_j f_k f_l^* f_m^* \;\to$$

$$\sum_{j,l}(1-\tfrac{1}{2}\delta_+-\tfrac{1}{2}\delta_-)([g(j,l,j,l)+g(j,l,l,j)]|a_j|^2|a_l|^2+g(j,-j,l,-l)a_j a_{-j}a_l^* a_{-l}^*). \quad \text{(D 9)}$$

For the calculation of the nonlinear damping coefficients, forcing terms and terms containing the slow spatial derivative ∇_X can (and will) be neglected. The first term in E_2, $(\rho/2)\int \mathrm{d}^2x(\partial_t\zeta_0)\phi_2$, gives a contribution $-\frac{1}{4}\sigma k^4\mathscr{B}_{jl}^{(i)}$ (cf. (A 49)–(A 50)) to $H_{jl}^{(i)}$ (defined in (61)). However, the term $(\rho/2)\int \mathrm{d}^2x(\partial_t\zeta_2)\phi_0$ yields the opposite result, and thus the two terms cancel each other. The contribution to $H_{jl}^{(i)}$, obtained from the term $(\rho/2)\int \mathrm{d}^2x(\partial_T\zeta_0)\phi_0$, is exactly the right-hand side of (64). The contributions resulting from the remaining 8 terms in E_2 are given in table 3. Using the relations (79)–(80), between d and b, it follows that the total contribution from the five kinetic terms equals the total contribution from the three potential terms in table 3. The resulting corrections, $h_{jl}^{(i)}$ $(i=1,2)$, to (64) are given in (89)–(90).

D.4. *Corrections to the energy from the forcing*

The forcing gives corrections to the average energy. For the term containing the slow time derivative, we have (still neglecting slow spatial derivatives)

$$\left\langle \frac{\rho}{2}\int \mathrm{d}^2x\,(\partial_T\zeta_0)\phi_0\right\rangle = -\frac{\rho}{2}\frac{\mathrm{i}\omega}{k}\sum_j [a_j\partial_T a_j^* - a_j^*\partial_T a_j]$$

$$= -\tfrac{1}{2}\sigma k^2\sum_j [a_j a_{-j}+a_j^* a_{-j}^*] - (\sigma k^2/\omega)[\sum_{j,l} T_{jl}^{(1)}|a_l|^2|a_j|^2 + \sum_{j,l} T_{jl}^{(2)} a_l a_{-l} a_j^* a_{-j}^*], \quad \text{(D 10)}$$

Term	$\times\sigma k^2(a_j a_{-j} + a_j^* a_{-j}^*)$
$\frac{1}{2}\rho(\partial_t\zeta_0)\phi_2$	$-\frac{1}{4}$
$\frac{1}{2}\rho(\partial_t\zeta_2)\phi_0$	$\frac{1}{4}$
$\sum$	0
$-2(\rho\omega^2/k)\cos(2\omega t)\zeta_0^2$	-1
$\sigma\nabla\zeta_0\cdot\nabla\zeta_2$	$\frac{1}{2}$
$\sum$	$-\frac{1}{2}$

TABLE 4. Contributing terms to the kinetic and potential energies arising from the forcing

where the dispersion relation, (12) has been used. The non-zero forcing corrections to the energy from the other terms in (D 8) are given in table 4.

D.5. *The Lagrangian*

Now, consider the Lagrangian

$$L = \int \mathrm{d}^2x\ [\mathscr{T} - \mathscr{V}], \tag{D 11}$$

for the driven system. Based on the results above (and table 4), we have

$$\langle L\rangle = \epsilon^2\frac{\rho\omega}{2k}\sum_j[-\mathrm{i}(a_j\partial_T a_j^* - a_j^*\partial_T a_j) + \omega(a_j a_{-j} + a_j^* a_{-j}^*)]. \tag{D 12}$$

Using (D 10), $\langle L\rangle$ can be written as follows:

$$\langle L\rangle = \epsilon^2(\sigma k^2/\omega)[-\mathrm{i}\sum_j(a_j\partial_T a_j^* - a_j^*\partial_T a_j) + \omega\sum_j(a_j a_{-j} + a_j^* a_{-j}^*) + \sum_{j,l} T_{jl}^{(1)}|a_l|^2|a_j|^2 + \sum_{j,l} T_{jl}^{(2)} a_l a_{-l} a_j^* a_{-j}^*]. \tag{D 13}$$

Returning to unscaled units ($\epsilon\nabla_X \to \nabla_\perp$, $\epsilon\partial_T \to \partial_t$, $\epsilon^{1/2}a_j \to a_j$), (86) follows.

Appendix E. Expansion of the energy dissipation function

In this Appendix we consider in detail the energy dissipation function D defined by (60).

E.1. *The dissipation function*

First, we rewrite (60) in terms of the surface fields. We have ($\boldsymbol{v} = \nabla\phi$)

$$\begin{aligned} D &= -\frac{\eta}{2}\int \mathrm{d}^3x\ (2\partial_i\partial_j\phi)^2 = -\frac{\eta}{2}\int \mathrm{d}^3x\ 2\partial_i(2\partial_j\phi\partial_i\partial_j\phi) \\ &= -\frac{\eta}{2}\int \mathrm{d}^3x\ 2\nabla^2(\partial_j\phi)^2 = -\frac{\eta}{2}\int \mathrm{d}^3x\ 2\nabla^2\partial_j(\phi\partial_j\phi) = -\frac{\eta}{2}\int \mathrm{d}^3x\ \nabla^4\phi^2. \end{aligned} \tag{E 1}$$

Above we have used the incompressibility condition $\nabla^2\phi = 0$.

Term	$\times\lvert a_l\rvert^2\lvert a_j\rvert^2$	$\times a_l a_{-l} a_j^* a_{-j}^*$	
D_2 $[\nu\sigma k^6]$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$	$\times 1$	$\times(1-\frac{1}{2}\delta_{jl}-\frac{1}{2}\delta_{-jl})$
$\frac{1}{2}\eta\partial_z^3\phi_1^2$	$4c_+b^2(1+c)$	0	0
$\eta\partial_z^3(\phi_0\phi_2)$	$-4d-2b-8(1-c)^2$	4	$-8-4c^2$
$\frac{1}{2}\eta\zeta_1\nabla^4(\phi_0^2)$	$-2d(1-c)^2+10+4c+2c^2$	-8	$2+2c^2$
$\eta\zeta_0\nabla^4(\phi_0\phi_1)$	$4(1+c+2c_+)^2b$	0	0
$\frac{1}{4}\eta\zeta_0^2\partial_z\nabla^4(\phi_0^2)$	$32+32c$	0	$-8+8c^2$
$\sum$	$4[b(b+9-\frac{3}{2}c-c^2)-3d+17-10c-9c^2]$	-4	$-14+6c^2$

TABLE 5. Contributing terms of $O(\epsilon^2)$ in the expansion of the energy dissipation function. All terms proportional to c vanish in the last column (use $l \to -l$)

Invoking the general expansion

$$\int \mathrm{d}^3x\, f(\boldsymbol{x},z) = \int \mathrm{d}^2x \int_{-\infty}^{\zeta} f(\boldsymbol{x},z)\,\mathrm{d}z$$
$$= \int \mathrm{d}^2x \,[\int_{-\infty}^{0} f(\boldsymbol{x},z)\,\mathrm{d}z + (\zeta+\tfrac{1}{2}\zeta^2\partial_z)f(\boldsymbol{x},0)], \tag{E 2}$$

a straightforward calculation yields

$$D = -\frac{\eta}{2}\int \mathrm{d}^2x\,[\partial_z^3\phi^2(x,0) + (\zeta+\tfrac{1}{2}\zeta^2\partial_z)\nabla^4\phi^2(x,0)]. \tag{E 3}$$

E.2. *The average dissipation*

Introducing the slow scales $\boldsymbol{X} = \epsilon\boldsymbol{x}$ and $T = \epsilon t$, and the expansion (A8) for $\boldsymbol{\Psi}$ as in Appendix D, we obtain

$$\langle D\rangle = -\epsilon\langle D_1\rangle - \epsilon^2\langle D_2\rangle, \tag{E 4}$$

where

$$D_1 = \frac{\eta}{2}\int \mathrm{d}^2x\,\partial_z^3\phi_0^2, \tag{E 5}$$

$$D_2 = \frac{\eta}{2}\int \mathrm{d}^2x\,[\partial_z^3\phi_1^2 + 2\partial_z^3(\phi_0\phi_2) + \zeta_1\nabla^4\phi_0^2 + 2\zeta_0\nabla^4(\phi_0\phi_1) + \tfrac{1}{2}\zeta_0^2\partial_z\nabla^4\phi_0^2] \tag{E 6}$$

(every term evaluated at $z = 0$). Based on the results for $\boldsymbol{\Psi}$ from Appendix A, and the 'reduction formula' (D 9), the average energy dissipation function $\langle D\rangle$ (equation (62)) can be derived. Again, we do not include the higher-order resonant corrections $(b_j,\ c_j)$ to the amplitude field in our analysis. Moreover, we have neglected terms containing the forcing or the slow spatial derivative ∇_x.

From (E 5) the result given in (95) for $D^{(0)}$ (defined in (62)) is obtained. The contributions resulting from the five terms in D_2 are listed in table 5. The resulting coefficients, $D_{jl}^{(i)}$ $(i = 1, 2)$, defined in (62), are given in (97)–(98).

E.3. *Forcing terms*

As an extra check of the dissipation formalism, we consider the forcing terms arising on both sides of the energy dissipation formula, (59). According to §D.4 and table 4, we note that the forcing gives the correction $-\sigma k^2\sum_j[a_j a_{-j} + a_j^* a_{-j}^*]$ to the average

energy $\langle E\rangle$. Therefore, it also gives corrections to the average energy dissipation $\langle \dot{E}\rangle$, namely

$$-\sigma k^2 \partial_T \sum_j [a_j a_{-j} + a_j^* a_{-j}^*] = 4\nu\sigma k^4 \sum_j [a_j a_{-j} + a_j^* a_{-j}^*] \tag{E 7}$$

($\partial_T a_j \to -2\nu k^2 a_j$). This term equals exactly the contribution to $\langle D_2\rangle$ obtained from the forcing term $(\mathrm{i}\omega/2k)\mathrm{e}^{kz}\sum_j a_{-j}^* f_j t_\omega + \text{c.c. in } \phi_2$ (cf. (A 52)),

$$\left\langle \frac{\eta}{2}\int \mathrm{d}^2x\ [2\partial_z^3(\phi_0\phi_2)]\right\rangle \to \frac{\eta}{2}(2k)^3\left(\frac{\omega}{k}\right)^2 \sum_j [a_j a_{-j} + a_j^* a_{-j}^*]$$
$$= 4\nu\sigma k^4 \sum_j [a_j a_{-j} + a_j^* a_{-j}^*]. \tag{E 8}$$

A second contribution to the average energy dissipation from the forcing comes from $\langle \dot{E}_1\rangle$,

$$\sum_j H^{(0)}\partial_T |a_j|^2 \to 2\mathrm{i}\sigma k^2\omega \sum_j [-a_j a_{-j} + a_j^* a_{-j}^*] \tag{E 9}$$

($\partial_T a_j \to \mathrm{i}\omega a_{-j}^*$). This result equals that obtained from the integral in the energy dissipation formula, (59). We have ($\rho f \to 4\sigma k^2\epsilon$),

$$\left\langle \int \mathrm{d}^2x\ [2\sigma k^2\omega_e \sin(\omega_e t)\zeta_0^2]\right\rangle = 2\mathrm{i}\sigma k^2\omega \sum_j \langle [t_{2\omega} - t_{-2\omega}][a_j a_{-j} t_{2\omega} + a_j^* a_{-j}^* t_{-2\omega}]\rangle$$
$$= 2\mathrm{i}\sigma k^2\omega \sum_j [-a_j a_{-j} + a_j^* a_{-j}^*]. \tag{E 10}$$

Appendix F. Newell–Whitehead expansion for standing waves

In this Appendix we show in details the steps in the Newell–Whitehead expansion of the set of coupled amplitude equations for $\boldsymbol{u} = (a_j, a_{-j}^*)$. The issue is to obtain a single amplitude equation for standing waves including spatial derivatives which was explicitly neglected in the Lagrangian description by Miles.

F.1. *The neutral solution*

A set of linearly unstable modes (standing waves) is obtained in a truncated linear stability analysis which neglects all nonlinear terms. From §5.1 we have

$$\boldsymbol{u}_0 = (1, -\mathrm{i})A_j \tag{F 1}$$

as the neutral solution in a Newell–Whitehead expansion.

F.2. *Scaling of variables*

We now scale our variables in terms of ϵ. We expand around $\Delta\omega(\boldsymbol{q}) = 0$ (cf. (108)), so the natural scaling is $\boldsymbol{X} = \epsilon^{1/2}\boldsymbol{x}$, $T = \epsilon t$, and

$$\boldsymbol{u} = (a_j, a_{-j}^*) = \epsilon^{1/2}\boldsymbol{u}_0 + \epsilon\boldsymbol{u}_1 + \epsilon^{3/2}\boldsymbol{u}_2 + \ldots. \tag{F 2}$$

When we apply this scaling, we obtain three equations at different orders,

$$\boldsymbol{L}_0\boldsymbol{u}_0 = 0, \tag{F 3}$$

$$\boldsymbol{L}_0\boldsymbol{u}_1 = -\boldsymbol{L}_1\boldsymbol{u}_0, \tag{F 4}$$

$$\boldsymbol{L}_0\boldsymbol{u}_2 = -\boldsymbol{L}_1\boldsymbol{u}_1 - (\partial_T + \boldsymbol{L}_2)\boldsymbol{u}_0 - \boldsymbol{N}, \tag{F 5}$$

where the linear operators $\boldsymbol{L}_i$ $(i = 0, 1, 2)$ are

$$\boldsymbol{L}_0 = \begin{pmatrix} \gamma^{(0)} & -\mathrm{i}\gamma^{(0)} \\ \mathrm{i}\gamma^{(0)} & \gamma^{(0)} \end{pmatrix}, \quad \boldsymbol{L}_1 = \begin{pmatrix} \mathscr{G}_j^{(1)} & 0 \\ 0 & -\mathscr{G}_j^{(1)} \end{pmatrix}, \quad \boldsymbol{L}_2 = \begin{pmatrix} \mathrm{i}\mathscr{G}_j^{(2)} & -\mathrm{i}\omega \\ \mathrm{i}\omega & -\mathrm{i}\mathscr{G}_j^{(2)} \end{pmatrix}, \tag{F 6}$$

and the vector $\boldsymbol{N}$ comprises all the nonlinear terms,

$$\boldsymbol{N} = \begin{pmatrix} \sum_l [\gamma(\theta_{jl}) - \mathrm{i}T(\theta_{jl})]|A_l|^2 A_j \\ -\mathrm{i}\sum_l [\gamma(\theta_{jl}) + \mathrm{i}T(\theta_{jl})]|A_l|^2 A_j \end{pmatrix}. \tag{F 7}$$

In (F 6), we have introduced the expansion

$$\mathscr{G}_j = \epsilon^{1/2}\mathscr{G}_j^{(1)} + \mathrm{i}\epsilon\mathscr{G}_j^{(2)}, \tag{F 8}$$

where

$$\mathscr{G}_j^{(1)} = c_1\hat{\boldsymbol{k}}_j \cdot \nabla_X, \quad \mathscr{G}_j^{(2)} = c_2(\hat{\boldsymbol{k}}_j \cdot \nabla_X)^2 - c_3\nabla_X^2. \tag{F 9}$$

Regarding the nonlinear terms, (F 7), we have used that

$$\begin{aligned} \sum_l [\gamma_{jl} \pm \mathrm{i}T_{jl}]|A_l|^2 A_j &= \sum_l [\gamma_{-jl} \pm \mathrm{i}T_{-jl}]|A_l|^2 A_j \\ &= \sum_l [\gamma(\theta_{jl}) \pm \mathrm{i}T(\theta_{jl})]|A_l|^2 A_j. \end{aligned} \tag{F 10}$$

F.3. *Solvability conditions*

In order to obtain the solvability conditions for the above system of equations, we note that

$$\boldsymbol{u}_l = (1, \mathrm{i}) \tag{F 11}$$

is a left eigenvector to $\boldsymbol{L}_0$ with $\boldsymbol{u}_l \cdot \boldsymbol{L}_0 = 0$. The solvability conditions are the requirement that left multiplication by $\boldsymbol{u}_l$ on the right-hand sides of (F 4) and (F 5) are 0 as well.

F.4. *The amplitude equation*

The solvability condition obtained from the $n = 1$ problem, (F 4), is trivially fulfilled. Calculating the right-hand side of (F 4), we obtain

$$\boldsymbol{L}_0\boldsymbol{u}_1 = \begin{pmatrix} -\mathscr{G}^{(1)}A_j \\ -\mathrm{i}\mathscr{G}^{(1)}A_j \end{pmatrix}. \tag{F 12}$$

For $\boldsymbol{u}_1$, we then obtain the solution

$$\boldsymbol{u}_1 = -\frac{1}{2\gamma^{(0)}}(\mathscr{G}_j^{(1)}A_j, i\mathscr{G}_j^{(1)}A_j). \tag{F 13}$$

Next, consider the $n = 2$ problem (F 5). We obtain

$$\boldsymbol{L}_0\boldsymbol{u}_2 = \begin{pmatrix} \dfrac{1}{2\gamma^{(0)}}(\mathscr{G}^{(1)})^2 A_j - \partial_T A_j - \mathrm{i}\mathscr{G}^{(2)}A_j + \omega A_j - \displaystyle\sum_l [\gamma(\theta_{jl}) - \mathrm{i}T(\theta_{jl})]|A_l|^2 A_j \\ -\mathrm{i}\dfrac{1}{2\gamma^{(0)}}(\mathscr{G}^{(1)})^2 A_j + \mathrm{i}\partial_T A_j - \mathrm{i}\omega A_j + \mathscr{G}^{(2)}A_j + \mathrm{i}\displaystyle\sum_l [\gamma(\theta_{jl}) + \mathrm{i}T(\theta_{jl})]|A_l|^2 A_j \end{pmatrix}. \tag{F 14}$$

From (F 11), we find the solvability condition

$$0 = \partial_T A_j - \omega A_j - \frac{1}{2\gamma^{(0)}}(\mathscr{G}^{(1)})^2 A_j + \sum_l \gamma(\theta_{jl})|A_l|^2 A_j. \tag{F 15}$$

Returning to unscaled units ($\epsilon^{1/2}\nabla_X \to \nabla_\perp$, $\epsilon\partial_T \to \partial_t$, $\epsilon^{1/2}A_j \to A_j$), the amplitude equation (110), follows.

REFERENCES

Bechhoefer, J., Ego, V., Manneville, S. & Johnson, B. 1995 *J. Fluid Mech.* **288**, 325.

Benjamin, T. B. & Ursell, F. 1954 *Proc R. Soc. Lond.* A **225**, 505.

Beyer, J. & Friedrich, R. 1995 *Phys. Rev.* E **51**, 1162.

Binks, D. & Water, W. van de 1997 *Phys. Rev. Lett.* **78**, 4043.

Bosch, E. & Water, W. van de 1993 *Phys. Rev. Lett.* **70**, 3420.

Bosch, E. G. T. 1995 An experimental investigation of Faraday waves and spatio-temporal chaos. PhD thesis, University of Eindhoven (unpublished).

Christiansen. B. 1993 Faraday waves and collective phenomena in coupled oscillators. PhD thesis, University of Copenhagen (unpublished).

Christiansen, B., Alstrøm, P. & Levinsen, M. 1992 *Phys. Rev. Lett.* **68**, 2157.

Christiansen, B., Alstrøm, P. & Levinsen, M. 1995 *J. Fluid Mech.* **291**, 323.

Ciliberto, S. & Gollub, J. P. 1984 *Phys. Rev. Lett.* **52**, 922.

Ciliberto, S. & Gollub, J. P. 1985 *J. Fluid Mech.* **158**, 381.

Craig, W. & Sulem, C. 1993 *J. Comput. Phys.* **108**, 73.

Cross, M. C. & Hohenberg, P. C. 1993 *Rev. Mod. Phys.* **65**, 851.

Douady, S. 1990 *J. Fluid Mech.* **221**, 383.

Douady, S. & Fauve, S. 1988 *Europhys. Lett.* **6**, 221.

Edwards, W. S. & Fauve, S. 1992 *C. R. Acad. Sci. Paris* II **315**, 417.

Edwards, W. S. & Fauve, S. 1993 *Phys. Rev.* E **47**, R788.

Edwards, W. S. & Fauve, S. 1994 *J. Fluid Mech.* **278**, 123.

Ezerskii, A. B., Korotin, P. I. & Rabinovich, M. I. 1985 *Pis'ma Zh. Eksp. Teor. Fiz.* **41**, 129 (*JETP Lett.* **41**, 157 (1985)).

Ezerskii, A. B., Rabinovich, M. I., Reutov, V. P. & Starobinets, I. M. 1986 *Zh. Eksp. Teor. Fiz.* **91**, 2070 (*Sov. Phys. JETP* **64**, 1228 (1986)).

Ezersky, A. B., Kiyashko, S. V., Matusov, P. A. & Rabinovich, M. I. 1994 *Europhys. Lett.* **26**, 183.

Faraday, M. 1831 *Phil. Trans. R. Soc. Lond.* **52**, 299.

Fauve, S., Kumar, K., Laroche, C., Beysens, D. & Garrobs, Y. 1992 *Phys. Rev. Lett.* **68**, 3160.

Kambe, T. & Umeki, M. 1990 *J. Fluid. Mech.* **212**, 373.

Kudrolli, A. & Gollub, G. P. 1996 *Physica* D **97**, 133.

Kumar, K. 1996 *Proc. R. Soc. Lond.* A **452**, 1113.

Kumar, K. & Tuckerman, L. 1994 *J. Fluid Mech.* **279**, 49.

Landau, L. D. & Lifshitz, E. M. 1976 *Course of Theoretical Physics; Vol. 1, Mechanics*, 3rd edn. Pergamon.

Landau, L. D. & Lifshitz, E. M. 1987 *Course of Theoretical Physics; Vol. 6, Fluid Mechanics*, 2nd edn. Pergamon.

Levin, B. V. & Trubnikov, B. A. 1986 *Pis'ma Zh. Eksp. Teor. Fiz.* **44**, 311 (*JETP Lett.* **44**, 399 (1986)).

Malomed, B. A., Nepomnyashchii, A. A., & Tribelskii, M. I. 1989 *Zh. Eksp. Teor. Fiz.* **96**, 684. (*Sov. Phys. JETP* **69**, 388 (1989)).

Manneville, P. 1990 *Dissipative Structures and Weak Turbulence*, 1st edn. Academic.

Milder, M. 1977 *J. Fluid Mech.* **83**, 159.

Miles, J. W. 1976 *J. Fluid Mech.* **75**, 419.

Miles, J. W. 1977 *J. Fluid Mech.* **83**, 153.

Miles, J. W. 1984 *J. Fluid Mech.* **146**, 285.

Miles, J. W. 1992 *J. Fluid Mech.* **245**, 485.

Miles, J. W. 1993 *J. Fluid Mech.* **248**, 671.
Miles, J. W. 1994 *J. Fluid Mech.* **269**, 353.
Miles, J. W. & Henderson, D. 1990 *Ann. Rev. Fluid Mech.* **22**, 143.
Milner, S. T. 1991 *J. Fluid Mech.* **225**, 81.
Müller, H. W. 1993 *Phys. Rev. Lett.* **71**, 3287.
Müller, H. W. 1994 *Phys. Rev.* E **49**, 1273.
Müller, H. W., Wittmer, H., Wagner, C., Albers, J. & Knorr, K. 1997 *Phys. Rev. Lett.* **78**, 2357
Newell, A. C. & Whitehead, J. A. 1969 *J. Fluid Mech.* **38**, 279.
Rayleigh, Lord 1883 *Phil. Mag.* **16**, 50.
Riecke, H. 1990 *Europhys. Lett.* **11**, 213.
Simonelli, F. & Gollub, J. P. 1989 *J. Fluid Mech.* **199**, 471.
Tufillaro, N. B., Ramshankar, R. & Gollub, J. P. 1989 *Phys. Rev. Lett.* **62**, 422.
Umeki, M. 1991 *J. Fluid Mech.* **227**, 161.
Umeki, M. 1996 *J. Phys. Soc. Japan* **65**, 84.
Umeki, M. & Kambe, T. 1989 *J. Phys. Soc. Japan* **58**, 140.
Zakharov, V. E. 1968 *Zh. Prikl. Mekh. Tekh. Fiz.* **2**, 86 (*J. Appl. Mech. Tech. Phys.* **9**, 190 (1968)).
Zakharov, V. E., L'vov, V. S. & Starobinets, S. S. 1971 *Sov. Phys. JETP* **32**, 656.
Zakharov, V. E., L'vov, V. S. & Falkovich, G. 1992 *Kolmogorov Spectra of Turbulence I, Wave Turbulence*. Springer.
Zhang, W. & Viñals, J. 1996 *Phys. Rev.* E **53**, R4283.
Zhang, W. & Viñals, J. 1997 *J. Fluid Mech.* **336**, 301.